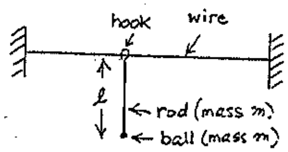
Spring 2000

MS/Ph.D. Comprehensive Examination - Spring 2000
Mechanics 600-1

A rod of mass m and length ℓ hangs vertically from a horizontal frictionless wire. Attached to the end of the rod is a small ball (assume it is a particle), also of mass m. The rod is free to slide along the wire, and to swing in the plane of the figure.



a) Find the location of the center of mass for rod plus ball, taking the hook as the origin of the coordinate system.

b) Find the moment of inertia about the center of mass.

c) Set up the Lagrange equations of motion for the linear and angular motion of the system. Use x of the center of mass and θ of the rod as coordinates.

d) From the equations of motion, find the frequency of small oscillations.

MS/Ph.D. Comprehensive Examination - Spring 2000
Mechanics 600-2

A moon of mass m travels in a circular orbit with angular velocity ω around a planet of mass M. Assume $m << M$. Assume the moon is a point particle, such that its rotation can be neglected, but the planet rotates about its axis with angular velocity Ω.

The axis of rotation is perpendicular to the plane of the moon's orbit.

Let I = moment of inertia of the planet about its axis.
Let D = distance from center of moon to that of the planet.

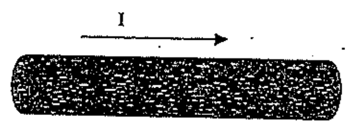
a.) Find expressions for the total angular momentum L of the system about its center of mass and the total energy E. (Eliminate D from both of these expressions).

b.) Generally, the two angular velocities are such that $\omega \neq \Omega$.

Suppose there is tidal friction that can reduce E, but conserves angular momentum.
Examine the behavior of E as a function of ω for fixed Ω
Show there is a range of initial conditions that eventually lead to $\omega = \Omega$ and a stable final configuration.

MS/Ph.D. Comprehensive Examination – Spring 2000
Electromagnetism 600-1

A very long conducting cylindrical wire of radius R carries a DC current I. The current is carried with equal density throughout the cross-section of the wire.



(a) Find the vector potential A both inside and outside the cylinder.

Hints: You may find it useful to reason by analogy.

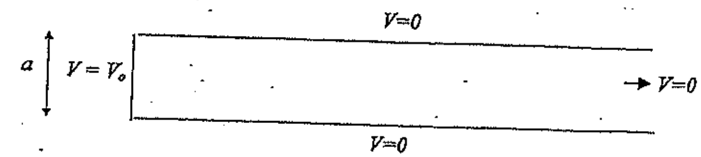
Useful facts: $\vec{A}(x) = \frac{\mu_o}{4\pi}\int \frac{\vec{J}(x')}{|x-x'|} d^3x'$ $\qquad \nabla^2 A_z = -\mu_o J_z$

(b) Calculate the magnetic field B from the vector potential.

(c) Calculate the magnetic field via Ampere's law and confirm that your result agrees with that found in part (b).

MS/Ph.D. Comprehensive Examination – Spring 2000
Electromagnetism 600-2

Two semi-infinite plates lie parallel to one another and are grounded. A third plate lies perpendicular to these plates at one end, separated by a thin layer of insulator, and is held at elevated potential $V = V_o$. The potential at $x = \infty$ between the two plates drops off to 0. The plates are shown in cross-section.



(a) Define a coordinate system (x,y) and find an expression for the potential between the plates V(x,y), using the following steps:

→ Write down Laplace's equation in two Cartesian dimensions and carry out the separation of variables explicitly using V(x,y) = X(x) Y(y).

→ Choose a sign for the separation constant and find solutions for X(x) and Y(y).

→ Fit the boundary conditions and find V(x,y) expressed as a summation, and write down an expression for evaluation of the coefficients.

(b) Write an expression for the surface charge density on the lower plate.

Consider an electromagnetic wave of angular frequency ω where $\vec{E} = E_0 \exp[i(\vec{K}\cdot\vec{r} - \omega t)]$ propagating through a neutral (collisionless) plasma of electrons and heavy ions ($\rho = 0$).

(a) What is the equation of motion of an electron due to $\vec{E}$.

(b) Find the current density induced by $\vec{E}$ (neglect interactions of electrons).

(c) Write down Maxwell's Equations and give the resulting differential equations for the spatial dependence of a wave of angular frequency ω in such a medium.

(d) From the equations above, derive the sufficient and necessary conditions

$$\rho_e < \frac{\epsilon_0 m_e \omega^2}{e^2}$$

allowing waves to propagate in this medium.

ρ_e is the electron number density.

Consider a gas of Cl_2 chloride, it consists of diatomic molecules.

(1) Calculate the gas specific heat, C_p, for a temperature range where the rotation, and translation degrees of freedom behave classically and the vibrational behave quantum mechanically.

(2) Derive the Debye specific heat for a low temperature monoatomic solid.

(3) Modify the Debye equation for the diatomic molecular chlorine solid.

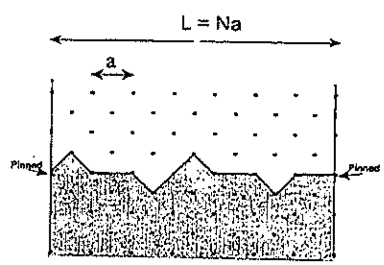
MS/Ph.D. Comprehensive Examination - Spring 2000
Statistical Mechanics 600-1

In this problem we study the properties of the surface of the model two-dimensional "liquid" shown in the figure below. The configurations of the surface are confined to a triangular lattice and the ends of the surface are pinned at the two points shown, which are a distance L apart. Assume that "overhangs" are forbidden so that, starting from one pin, every step along the surface is taken towards the other pin. Now consider the length of the surface. Take the zero-energy configuration to be the flat surface (of length L) and the energy required to increase the length by one lattice spacing a to be ϵ.

a) Write out the combined first and second laws of thermodynamics as applied to this system, ignoring pV work.

b) Use the microcanonical ensemble to calculate the length of the surface as a function of temperature, $l(T)$.

c) Similarly, calculate the (one-dimensional) surface tension as a function of temperature $\sigma(T)$ and show that it is properly an intensive variable. Is this a good model for the surface tension of a liquid?



MS/Ph.D. Comprehensive Examination - Spring 2000
Quantum Mechanics 600-1

Consider a particle in a one-dimensional box (with walls at $x=0$ and $x=a$) in its ground state. Suddenly, the walls of the box are removed. Find the probability $|a(p)|^2\,dp$ that the particle has momentum between p and $p+dp$. (Hint: consider the free-particle eigenfunctions, and use these to expand the initial wave function.)

Consider a particle of spin 1 which has the normalized S_z-eigenfunctions $|1,1>$, $|1,0>$, $|1,-1>$.

In this representation, $S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $S_y = \frac{i\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$.

a) Consider the state function $|\psi>$ which is an eigenfunction of the x-component of spin with eigenvalue $+\hbar$:

$$S_x|\psi> = +\hbar|\psi>.$$

Since the S_z-eigenfunctions form a complete basis, we can write $|\psi>$ as a linear combination: $|\psi> = a|1,1> + b|1,0> + c|1,-1>$.

Write the above eigenvalue equation in matrix form, and find the coefficients a, b, c which solve the equation. (Be sure that $|\psi>$ is normalized.)

b) Find the expectation values of S_x and of S_y for a spin-1 particle which is in the state $|\psi> = \frac{1}{\sqrt{3}}|1,1> - \frac{1}{\sqrt{3}}|1,0> + \frac{1}{\sqrt{3}}|1,-1>$.

a) Derive the Schwarz inequality

$$\langle f|f\rangle \langle g|g\rangle \geq |\langle f|g\rangle|^2 = \tfrac{1}{4}(\langle f|g\rangle + \langle g|f\rangle)^2$$

by considering the quantity

$$I = \langle(\lambda f + g)|(\lambda f + g)\rangle$$

where f and g are general functions of position and λ is an arbitrary complex number.

[Hint: Since λ is arbitrary, choose that value required to complete the proof.]

b) When the system is in the state ψ, the observables represented by the operators $\hat{A}$ and $\hat{B}$ have expectation values $\bar{a}$ and $\bar{b}$, respectively, with uncertainties $(\Delta a)^2$ and $(\Delta b)^2$, respectively, where $(\Delta a)^2 = \langle(\hat{A} - \bar{a})^2\rangle$ and $(\Delta b)^2 = \langle(\hat{B} - \bar{b})^2\rangle$. Use the Schwarz inequality with $f = (\hat{A} - \bar{a})\psi$ and $g = i(\hat{B} - \bar{b})\psi$ to show that

$$(\Delta a)^2(\Delta b)^2 \geq \left[\frac{i}{2}\overline{[\hat{A},\hat{B}]}\right]^2$$

where $\overline{[\hat{A},\hat{B}]}$ is the expectation value of the commutator of $\hat{A}$ and $\hat{B}$.

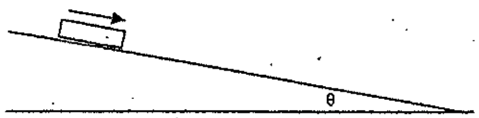
c) Show that for the observables position and linear momentum, this gives

$$\Delta p_x \Delta x \geq \hbar/2$$

d) Show that if $\Delta p_x \Delta x = \hbar/2$, then ψ is a Gaussian.

[Hint: Derive a differential equation for $\psi(x)$. For simplicity, you may assume $\bar{p}_x = \bar{x} = 0$ here.]

A sled of mass m slides down a snowy hill at an angle θ to the horizontal, as shown.



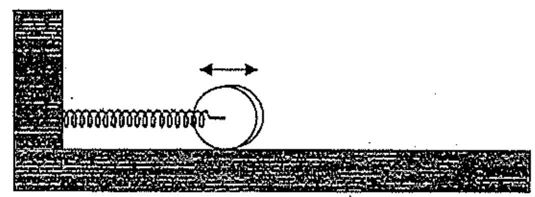
The sled's motion is affected by a drag force of the form $F_{drag} = -b\,v$.

(a) Find the sled's terminal velocity.

(b) If the sled starts from rest at time t=0, find the velocity as a function of time.

(c) After the sled approximately reaches its terminal velocity, how much energy does it lose per unit time due to the drag force? Where does this energy go? If the sled is losing energy, why doesn't it slow down further?

A harmonic oscillator consists of a mass spring system, where the mass is a solid cylindrical wheel of mass M and radius R that rolls without slipping on a horizontal surface, as shown. The spring is attached to an axle passing through the center of the wheel.



(a) Write down expressions for the **kinetic and potential energies** in terms of the wheel's horizontal displacement and velocity, taking into account the constraint of rolling without slipping. (Hint: what is the relationship between translational and angular position?) Assume the wheel's moment of inertia $I = (^1/_2)\,M R^2$.

(b) Construct the Lagrangian and find the equation of motion for the horizontal displacement of the wheel.

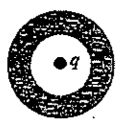
(c) The wheel is displaced slightly and released. What is the angular frequency of its oscillation?

(d) If the solid wheel is replaced by a bicycle wheel, whose mass is concentrated mostly near the outer rim, will the frequency of oscillation be higher or lower?

(e) In the limit that the moment of inertia I is zero, show that your answer to part (c) gives the result you'd expect for a simple harmonic oscillator.

MS Comprehensive Examination – Spring 2000
Electromagnetism 500-1

A point charge q sits at the center of a thick-walled **dielectric spherical shell** of inner radius R_1 and outer radius R_2, as shown.



(a) Find the fields D(r), E(r), and the polarization P(r).

(b) Evaluate the bound charge density in the volume of the sphere and on its inner and outer surfaces.

Useful facts:

$\nabla \bullet D = \rho \qquad D = \varepsilon E = \varepsilon_0 E + P$

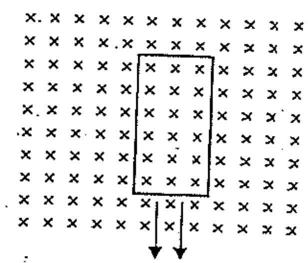
Bound charge densities: $\rho_b = \nabla \bullet \mathbf{P} \quad \sigma_b = \mathbf{P} \bullet \mathbf{n}$

Boundary conditions: $(\mathbf{D}_2 - \mathbf{D}_1) \bullet \mathbf{n}_{21} = 0 \quad (\mathbf{E}_2 - \mathbf{E}_1) \times \mathbf{n}_{21} = 0$

MS Comprehensive Examination – Spring 2000
Electromagnetism 500-2

A rectangular conducting loop of size $a \times b$ falls downward through a region, or window, containing a uniform magnetic field **B** pointing into the page as shown. Outside of this window the magnetic field is zero. The loop has resistance R and mass M.

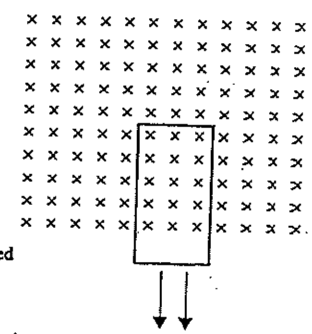
(a) While the loop is entirely inside the window, is any current induced in the loop as it falls? Explain your answer.



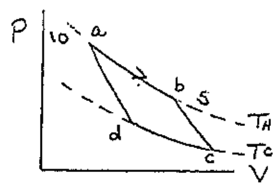
(b) The loop has fallen partly out of the window. If the loop falls at velocity v, what current is induced in the loop, and does it go clockwise or counter-clockwise? Explain your reasoning.

(c) Assuming the loop remains partially in the window, find its terminal velocity.

(d) When the loop closely approaches its terminal velocity, how much energy per unit time is dissipated in the loop as heat?



A heat engine contains 5 mol of an ideal gas having $\gamma = C_P/C_V =$ 1.5. It undergoes a Carnot cycle, operating between reservoirs of temperature $T_H = 400$ K and $T_C = 300$ K. The initial pressure (point a) is 10 atm, and during the initial isothermal expansion (ab) the pressure drops to 5 atm.



$R = 8.31$ J/mol-K

ab, cd: isothermal
bc, da: adiabatic

a) Make a table showing the heat absorbed, work done, and change in internal energy during each step.

b) Find the total heat absorbed and the total work done during the cycle.

c) Calculate the thermodynamic efficiency from the above results.

d) What is the significance of the area enclosed by the path on the P-V plot?

e) For a Carnot cycle, sketch a graph of entropy S versus temperature T. Label the points corresponding to a, b, c, d in the figure above.

f) What is the significance of the area enclosed by the path on the S-versus-T plot?

Consider two methods of heating a house by burning natural gas.

1. Burn gas at the house in a furnace.
2. Burn the gas at an electric utility, and use the generated electricity to run a heat pump at the house.

Make the following assumptions for method 2:

- The electric generator at the utility operates as a Carnot cycle between 527°C and 177°C. Assume there are no energy losses due to friction.
- There is no energy loss in the power lines.
- The electric motor of the heat pump converts electrical energy to work with 100% efficiency.
- The heat pump utilizes a Carnot cycle operating between an outside temperature 273 K and an inside temperature of 300 K.

a) Calculate the efficiency of the power plant for converting the chemical energy of gas to electrical energy.

b) Calculate the thermodynamic efficiency of the heat pump. (Hint: this efficiency is defined differently from that of an engine, and may be greater than one.)

c) What is the overall efficiency of method 2 for converting the energy of natural gas to heat?

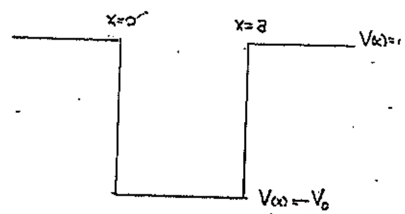
d) Is it possible to obtain the same overall efficiency using method 1? Explain.

Consider a 1-D square-well potential

$V(x) = 0 \qquad x < 0$

$V(x) = -V_0 \qquad 0 < x < \alpha$

$V(x) = 0 \qquad x > \alpha$



where V_0 is positive. If a particle of mass m is incident from the left with a positive energy $E < |V_0|$, derive an expression giving the probability for transmission through the potential. For what value of energy E is the probability unity?

Consider $y = y_1 + y_2$, which is the superposition of the two waves

$y_1 = \sin(kx - \omega x)$, and
$y_2 = \sin((k + dk)x - (\omega + d\omega)t)$,

where $dk << k$ and $d\omega << \omega$.

a) Show that, at any fixed time, $y(x)$ has the appearance of a chain of wave packets; that is, a short wavelength (λ_S) wave whose amplitude is modulated by a wave of longer wavelength (λ_L). Obtain expressions for λ_S and λ_L.

b) By considering the motion of the nodes of $y(x, t)$, show that the phase velocity and the group velocity are given by $v_p = \omega/k$ and $v_g = d\omega/dk$, respectively.

c) Show that the classical particle velocity is equal to the group velocity of the de Broglie wave.

d) Show that the relativistic particle velocity is also equal to the group velocity of the de Broglie wave.

CUA

THE CATHOLIC UNIVERSITY OF AMERICA

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200 Hannan Hall
Washington, D.C. 20064
202-319-5315
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MS Comprehensive Examination
Physics Department

Spring 2000

Thursday, March 16, and Friday, March 17, 2000

Room 231 - Conference Room

GENERAL INSTRUCTIONS:

This examination is divided into four sections as follows:

<u>Thursday, March 16, 2000</u>	
9:00 a.m. - 12:00 Noon	Classical Mechanics - 2 questions
1:00 P.M. - 5:00 P.M.	E & M - 2 questions
<u>Friday, March 17, 2000</u>	
9:00 a.m. - 12:00 Noon	Thermodynamics/Stat. Mech. - 2 questions
1:00 P.M. - 5:00 P.M.	Modern Physics/Quantum Mech. - 2 questions

<u>DO EACH PROBLEM IN A SEPARATE BLUE BOOK</u>

PUT YOUR NAME ON EACH BOOK

LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS
FOR EXAMPLE: Mechanics 600-1 or Electromagnetism 500-2

YOU <u>MUST</u> DO TWO (AND ONLY TWO) OF THE QUESTIONS IN EACH AREA.

OF THE TOTAL OF EIGHT QUESTIONS THAT ARE DONE, A <u>MINIMUM</u> OF TWO MUST BE FROM THE 600-LEVEL QUESTIONS.

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Ph.D. Comprehensive Examination

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FOR EXAMPLE: Mechanics 600-1

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Fax 202-319-4448

Ph.D/MS Comprehensive Examination

Spring 2000

Thursday, March 16, and Friday March 17, 2000

THE WRITTEN EXAM WILL BE HELD IN ROOM 133 HANNAN HALL

THE EXAM WILL BEGIN AT 9:00 AM ON MARCH 16, AND CONTINUE THRU 5:00 PM FRIDAY, MARCH 17, 2000

THE ORAL PORTION OF THE PH.D. COMPREHENSIVE EXAM WILL BEGIN THE FOLLOWING WEEK STARTING WEDNESDAY, MARCH 22, 2000.

PLEASE CALL (202) 319-5315) OR STOP BY THE PHYSICS DEPARTMENT OFFICE ON FRIDAY, (3/17) OR MONDAY, (3/20) TO FIND OUT YOUR SCHEDULED TIME AND ROOM ASSIGNMENT FOR THE ORAL EXAM.

Consider a gas of Cl_2 chloride, it consists of diatomic molecules.

(1) Calculate the gas specific heat, C_p, for a temperature range where the rotation, and translation degrees of freedom behave classically and the vibrational behave quantum mechanically.

(2) Derive the Debye specific heat for a low temperature monoatomic solid.

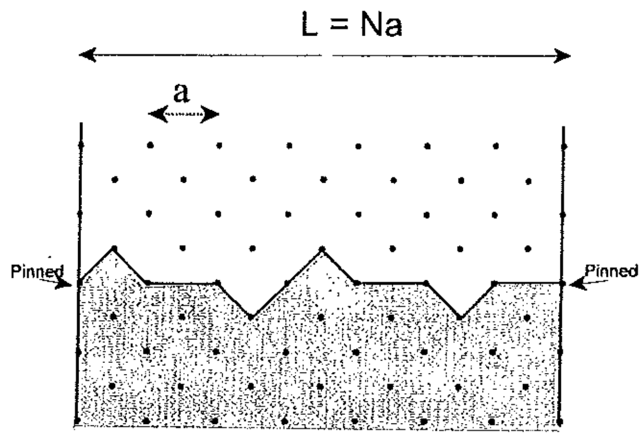
(3) Modify the Debye equation for the diatomic molecular chlorine solid.

In this problem we study the properties of the surface of the model two-dimensional "liquid" shown in the figure below. The configurations of the surface are confined to a triangular lattice and the ends of the surface are pinned at the two points shown, which are a distance L apart. Assume that "overhangs" are forbidden so that, starting from one pin, every step along the surface is taken towards the other pin. Now consider the length of the surface. Take the zero-energy configuration to be the flat surface (of length L) and the energy required to increase the length by one lattice spacing a to be ϵ.

a) Write out the combined first and second laws of thermodynamics as applied to this system, ignoring pV work.

b) Use the microcanonical ensemble to calculate the length of the surface as a function of temperature, $l(T)$.

c) Similarly, calculate the (one-dimensional) surface tension as a function of temperature $\sigma(T)$ and show that it is properly an intensive variable. Is this a good model for the surface tension of a liquid?



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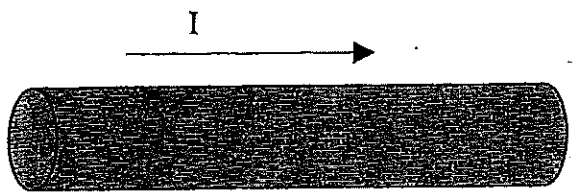

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Hints: You may find it useful to reason by analogy.

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(b) Calculate the magnetic field B from the vector potential.

(c) Calculate the magnetic field via Ampere's law and confirm that your result agrees with that found in part (b).

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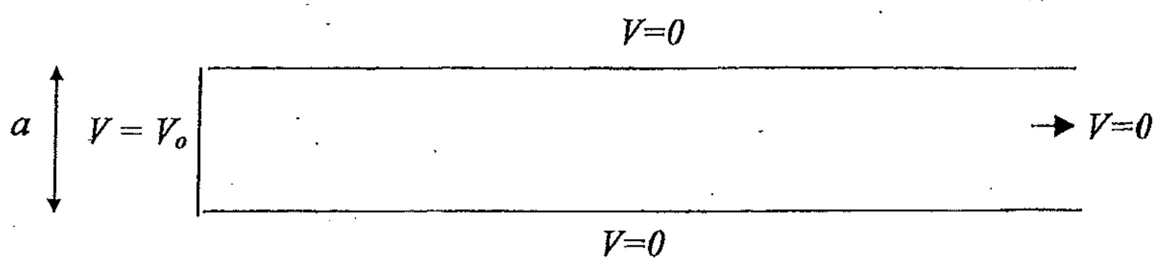

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$$\rho_e < \frac{\epsilon_0 m_e \omega^2}{e^2}$$

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ρ_e is the electron number density.

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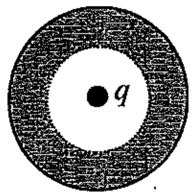

(a) Find the fields D(r), E(r), and the polarization P(r).

(b) Evaluate the bound charge density in the volume of the sphere and on its inner and outer surfaces.

Useful facts:

$\nabla \bullet D = \rho$ $\qquad$ $D = \varepsilon E = \varepsilon_o E + P$

Bound charge densities: $\rho_b = \nabla \bullet \mathbf{P}$ $\quad$ $\sigma_b = \mathbf{P} \bullet \mathbf{n}$

Boundary conditions: $(\mathbf{D}_2 - \mathbf{D}_1) \bullet \mathbf{n}_{21} = 0$ $\quad$ $(\mathbf{E}_2 - \mathbf{E}_1) \times \mathbf{n}_{21} = 0$

A rectangular conducting loop of size $a \times b$ falls downward through a region, or window, containing a uniform magnetic field **B** pointing into the page as shown. Outside of this window the magnetic field is zero. The loop has resistance R and mass M.

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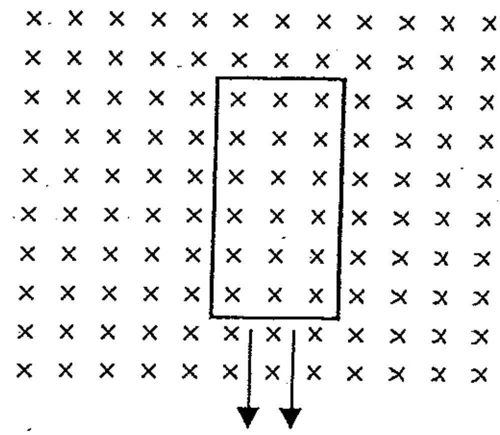

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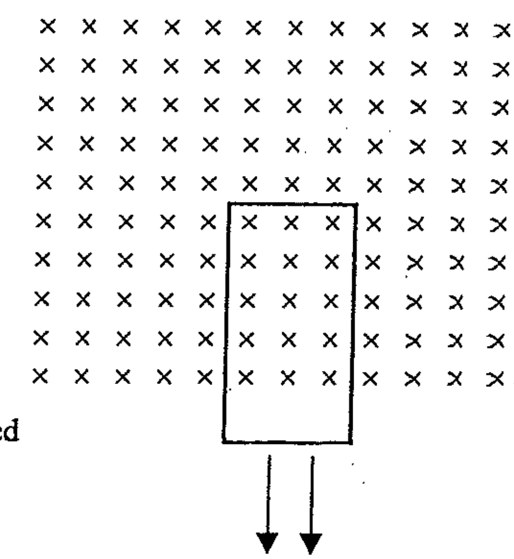
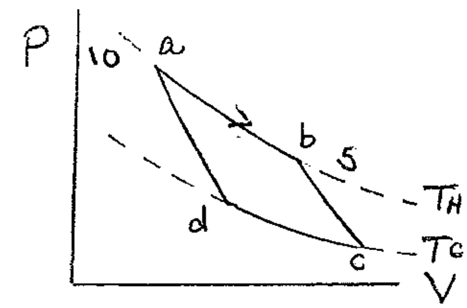
(c) Assuming the loop remains partially in the window, find its terminal velocity.

(d) When the loop closely approaches its terminal velocity, how much energy per unit time is dissipated in the loop as heat?



A heat engine contains 5 mol of an ideal gas having $\gamma = C_P/C_V =$ 1.5. It undergoes a Carnot cycle, operating between reservoirs of temperature $T_H = 400$ K and $T_C = 300$ K. The initial pressure (point *a*) is 10 atm, and during the initial isothermal expansion (*ab*) the pressure drops to 5 atm.



$R = 8.31$ J/mol-K

ab, cd: isothermal
bc, da: adiabatic

a) Make a table showing the heat absorbed, work done, and change in internal energy during each step.

b) Find the total heat absorbed and the total work done during the cycle.

c) Calculate the thermodynamic efficiency from the above results.

d) What is the significance of the area enclosed by the path on the *P-V* plot?

e) For a Carnot cycle, sketch a graph of entropy *S* versus temperature *T*. Label the points corresponding to *a, b, c, d* in the figure above.

f) What is the significance of the area enclosed by the path on the *S*-versus-*T* plot?

Consider two methods of heating a house by burning natural gas.

1. Burn gas at the house in a furnace.
2. Burn the gas at an electric utility, and use the generated electricity to run a heat pump at the house.

Make the following assumptions for method 2:

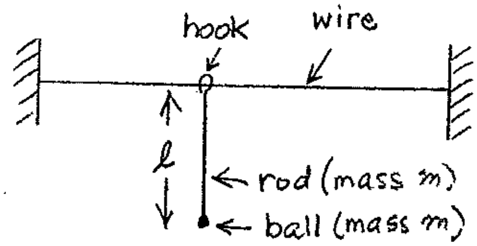
- The electric generator at the utility operates as a Carnot cycle between 527°C and 177°C. Assume there are no energy losses due to friction.
- There is no energy loss in the power lines.
- The electric motor of the heat pump converts electrical energy to work with 100% efficiency.
- The heat pump utilizes a Carnot cycle operating between an outside temperature 273 K and an inside temperature of 300 K.

a) Calculate the efficiency of the power plant for converting the chemical energy of gas to electrical energy.

b) Calculate the thermodynamic efficiency of the heat pump. (Hint: this efficiency is defined differently from that of an engine, and may be greater than one.)

c) What is the overall efficiency of method 2 for converting the energy of natural gas to heat?

d) Is it possible to obtain the same overall efficiency using method 1? Explain.



A rod of mass m and length ℓ hangs vertically from a horizontal frictionless wire. Attached to the end of the rod is a small ball (assume it is a particle), also of mass m. The rod is free to slide along the wire, and to swing in the plane of the figure.

a) Find the location of the center of mass for rod plus ball, taking the hook as the origin of the coordinate system.

b) Find the moment of inertia about the center of mass.

c) Set up the Lagrange equations of motion for the linear and angular motion of the system. Use x of the center of mass and θ of the rod as coordinates.

d) From the equations of motion, find the frequency of small oscillations.

A moon of mass m travels in a circular orbit with angular velocity ω around a planet of mass M. Assume $m << M$. Assume the moon is a point particle, such that its rotation can be neglected, but the planet rotates about its axis with angular velocity Ω.

The axis of rotation is perpendicular to the plane of the moon's orbit.

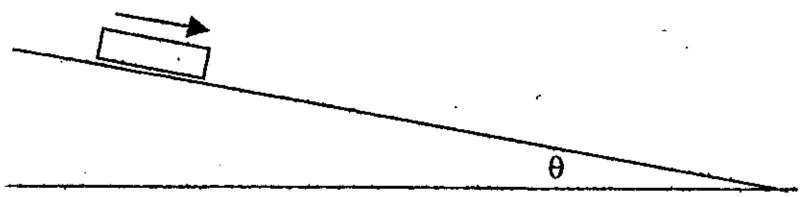
Let I = moment of inertia of the planet about its axis.
Let D = distance from center of moon to that of the planet.

a.) Find expressions for the total angular momentum L of the system about its center of mass and the total energy E. (Eliminate D from both of these expressions).

b.) Generally, the two angular velocities are such that $\omega \neq \Omega$.

Suppose there is tidal friction that can reduce E, but conserves angular momentum. Examine the behavior of E as a function of ω for fixed Ω
Show there is a range of initial conditions that eventually lead to $\omega = \Omega$ and a stable final configuration.

A sled of mass m slides down a snowy hill at an angle θ to the horizontal, as shown.



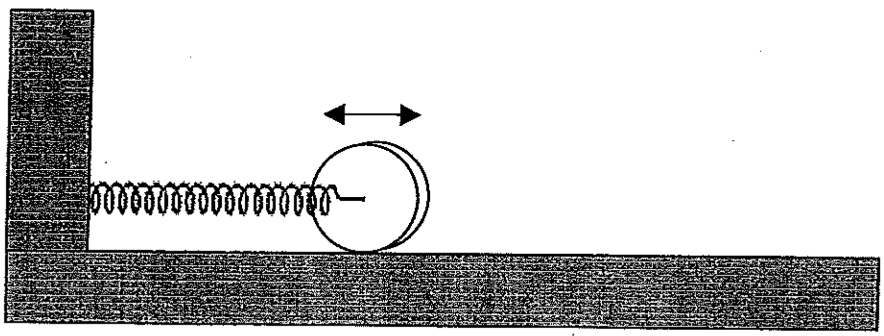
The sled's motion is affected by a drag force of the form $\boldsymbol{F}_{drag} = -b\,\boldsymbol{v}$.

(a) Find the sled's terminal velocity.

(b) If the sled starts from rest at time t=0, find the velocity as a function of time.

(c) After the sled approximately reaches its terminal velocity, how much energy does it lose per unit time due to the drag force? Where does this energy go? If the sled is losing energy, why doesn't it slow down further?

A harmonic oscillator consists of a mass spring system, where the mass is a solid cylindrical wheel of mass M and radius R that rolls without slipping on a horizontal surface, as shown. The spring is attached to an axle passing through the center of the wheel.



(a) Write down expressions for the **kinetic and potential energies** in terms of the wheel's horizontal displacement and velocity, taking into account the constraint of rolling without slipping. (Hint: what is the relationship between translational and angular position?) Assume the wheel's moment of inertia $I = (^1/_2)\, M R^2$.

(b) Construct the Lagrangian and find the equation of motion for the horizontal displacement of the wheel.

(c) The wheel is displaced slightly and released. What is the angular frequency of its oscillation?

(d) If the solid wheel is replaced by a bicycle wheel, whose mass is concentrated mostly near the outer rim, will the frequency of oscillation be higher or lower?

(e) In the limit that the moment of inertia I is zero, show that your answer to part (c) gives the result you'd expect for a simple harmonic oscillator.

Consider a particle in a one-dimensional box (with walls at $x=0$ and $x=a$) in its ground state. Suddenly, the walls of the box are removed. Find the probability $|a(p)|^2\, dp$ that the particle has momentum between p and $p+dp$. (Hint: consider the free-particle eigenfunctions, and use these to expand the initial wave function.)

Consider a particle of spin 1 which has the normalized S_z-eigenfunctions |1,1>, |1,0>, |1,-1>.

In this representation, $S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $S_y = \frac{i\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$.

a) Consider the state function $|\psi\rangle$ which is an eigenfunction of the x-component of spin with eigenvalue $+\hbar$:

$$S_x|\psi\rangle = +\hbar|\psi\rangle.$$

Since the S_z-eigenfunctions form a complete basis, we can write $|\psi\rangle$ as a linear combination: $|\psi\rangle = a|1,1\rangle + b|1,0\rangle + c|1,-1\rangle$.

Write the above eigenvalue equation in matrix form, and find the coefficients a, b, c which solve the equation. (Be sure that $|\psi\rangle$ is normalized.)

b) Find the expectation values of S_x and of S_y for a spin-1 particle which is in the state $|\psi\rangle = \frac{1}{\sqrt{3}}|1,1\rangle - \frac{1}{\sqrt{3}}|1,0\rangle + \frac{1}{\sqrt{3}}|1,-1\rangle$.

a) Derive the Schwarz inequality

$$\langle f|f\rangle \langle g|g\rangle \geq |\langle f|g\rangle|^2 = ¼\,(\langle f|g\rangle + \langle g|f\rangle)^2$$

by considering the quantity

$$I = \langle(\lambda f + g)|\,(\lambda f + g)\rangle$$

where f and g are general functions of position and λ is an arbitrary complex number.

[Hint: Since λ is arbitrary, choose that value required to complete the proof.]

b) When the system is in the state ψ, the observables represented by the operators $\hat{A}$ and $\hat{B}$ have expectation values $\bar{a}$ and $\bar{b}$, respectively, with uncertainties $(\Delta a)^2$ and $(\Delta b)^2$, respectively, where $(\Delta a)^2 \equiv \langle(\hat{A} - \bar{a})^2\rangle$ and $(\Delta b)^2 \equiv \langle(\hat{B} - \bar{b})^2\rangle$. Use the Schwarz inequality with $f = (\hat{A} - \bar{a})\psi$ and $g = i(\hat{B} - \bar{b})\psi$ to show that

$$(\Delta a)^2 (\Delta b)^2 \geq \left[\frac{i}{2}\overline{[\hat{A},\hat{B}]}\right]^2$$

where $\overline{[\hat{A},\hat{B}]}$ is the expectation value of the commutator of $\hat{A}$ and $\hat{B}$.

c) Show that for the observables position and linear momentum, this gives

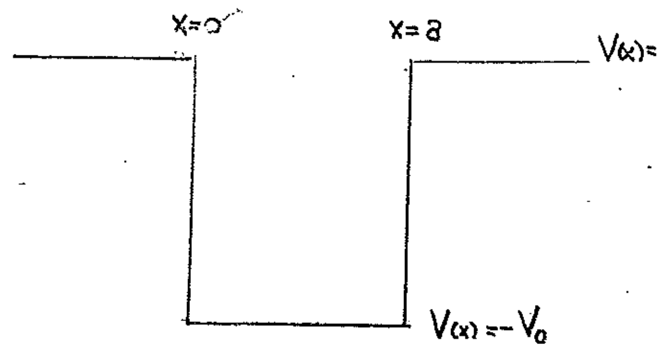
$$\Delta p_x \Delta x \geq \hbar/2$$

d) Show that if $\Delta p_x \Delta x = \hbar/2$, then ψ is a Gaussian.

[Hint: Derive a differential equation for $\psi(x)$. For simplicity, you may assume $\bar{p}_x = \bar{x} = 0$ here.]

MS Comprehensive Examination - Spring 2000
Modern Physics 500-1

Consider a 1-D square-well potential



$$V(x) = 0 \qquad x < 0$$

$$V(x) = -V_0 \qquad 0 < x < \alpha$$

$$V(x) = 0 \qquad x > \alpha$$

where V_0 is positive. If a particle of mass m is incident from the left with a positive energy $E < |V_0|$, derive an expression giving the probability for transmission through the potential. For what value of energy E is the probability unity?

Consider $y = y_1 + y_2$, which is the superposition of the two waves

$$y_1 = \sin(kx - \omega t), \text{ and}$$
$$y_2 = \sin((k + dk)x - (\omega + d\omega)t),$$

where $dk << k$ and $d\omega << \omega$.

a) Show that, at any fixed time, $y(x)$ has the appearance of a chain of wave packets; that is, a short wavelength (λ_S) wave whose amplitude is modulated by a wave of longer wavelength (λ_L). Obtain expressions for λ_S and λ_L.

b) By considering the motion of the nodes of $y(x, t)$, show that the phase velocity and the group velocity are given by $v_p = \omega/k$ and $v_g = d\omega/dk$, respectively.

c) Show that the classical particle velocity is equal to the group velocity of the de Broglie wave.

d) Show that the relativistic particle velocity is also equal to the group velocity of the de Broglie wave.