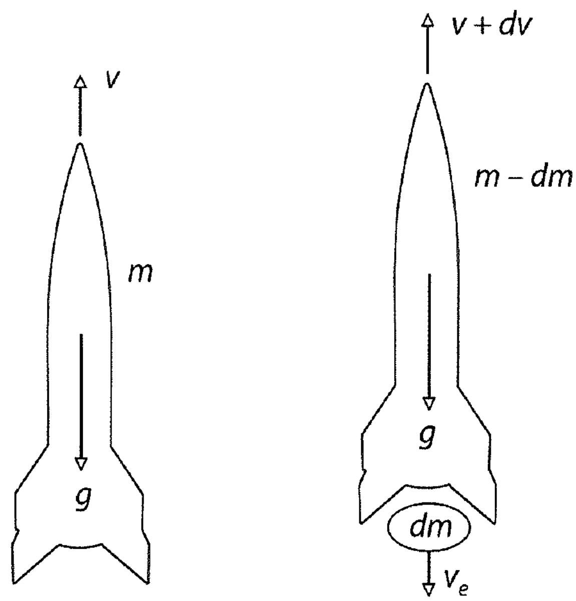
A rocket has an initial mass m_0 at time t=0. The rocket follows a vertical path after launch and consumes its fuel at a constant rate.



a) Assuming no external forces, show that the change in velocity ("delta-v") can be written as

$$\Delta v = v_e \ln\left(\frac{m_0}{m_f}\right),$$

where v_e is the exhaust velocity of the rocket fuel relative to the rocket, m_0 the initial mass of the rocket and m_f the final mass after the burn.

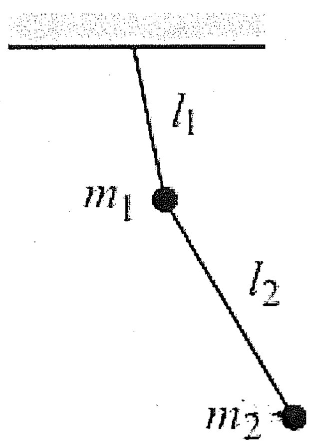
b) Assuming that the rocket launches vertically from the Earth, show that the velocity of the rocket at some arbitrary time t after liftoff is given by

$$v(t) = v_0 - v_e \ln\left(1 + \frac{t}{m_0}\frac{dm}{dt}\right) - gt$$

(Hint: Disregard the effect of drag (which in any case is small compared to gravity) and remember that the "mass-flow" is negative).

Consider a double pendulum made of two masses, m_1 and m_2, and two rods of lengths l_1 and l_2.

a) Find the equations of motion.



b) For small oscillations, find the normal modes and their frequencies for the special case $l_1 = l_2$ (and consider the cases $m_1 = m_2$, $m_1 >> m_2$, and $m_1 << m_2$).

c) Do the same for the special case $m_1 = m_2$ (and consider the cases $l_1 = l_2$, $l_1 >> l_2$, and $l_1 << l_2$).

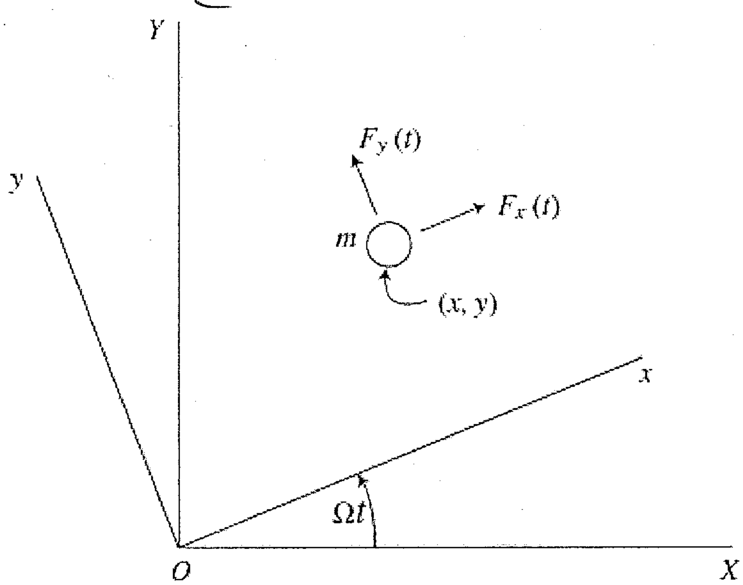
The Cartesian xy frame rotates at a constant rate Ω relative to the inertial XY frame (both frames share the origin). A particle P of mass m moves in the xy-plane under the action of arbitrary force components $F_x(t)$ and $F_y(t)$ – see figure below. Note that the force components are measured relative to the rotating frame. Also, assume that conservative forces are absent.

a) Using (x,y) for the coordinates of the particle in the rotating frame, obtain an expression for the kinetic energy of the particle in terms of its velocity relative to the rotating frame and the angular velocity Ω (and its mass). (40)

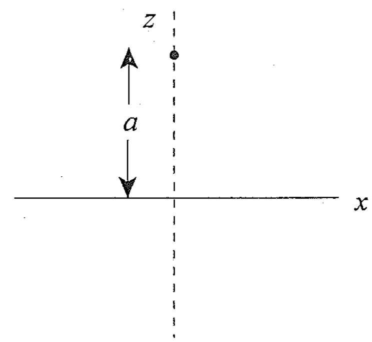
suggestion (AS)

b) Suppose a system is described by generalized coordinates $\{q_r\}$, derive or write down the general Lagrange equations of motion for such a system in the presence of non-conservative forces. These equations will involve the generalized forces $\{Q_r\}$. Assume that conservative forces are absent. (20)

c) Using (x,y) as the generalized coordinates, obtain the equations of motion for particle P. (40)



A positive point charge $+q$ is located on the z-axis, a distance a above a grounded conducting plane at z=0.



a) Using the method of images, find the exact potential $V(x,z)$ at any point in the xz plane.

b) Find the x- and z-components of the electric field at any point in the xz plane.

c) Find the magnitude and direction of the electric field at any point $(x,y,0)$ just above the conducting plane.

d) Find the surface charge density $\sigma(x,y)$ on the conducting plane.

e) Integrate $\sigma(x,y)$ to find the total induced charge on the conducting plane.

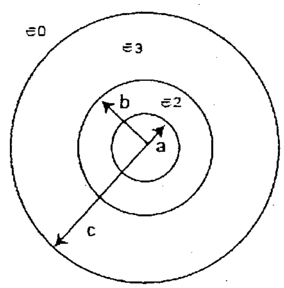
Consider two infinitely long concentric cylinders, as shown in the figure below. The inner cylinder of radius a is a conductor with linear charge density $\lambda_1 > 0$. The second cylinder with inner radius b and outer radius c consists of a material with permittivity ε_3 and is uniformly charged with linear charge density $\lambda_3 < 0$ ($\lambda_1 > |\lambda_3|$). The space between the two cylinders (i.e. $a < r < b$) is filled with a medium of permittivity ε_2. The medium outside the outer cylinder possesses the permittivity ε_0.

A) Compute the electric field $\vec{E}$ and displacement $\vec{D}$ in the different regions:

1. $r < a$
2. $a < r < b$
3. $b < r < c$
4. $c < r$

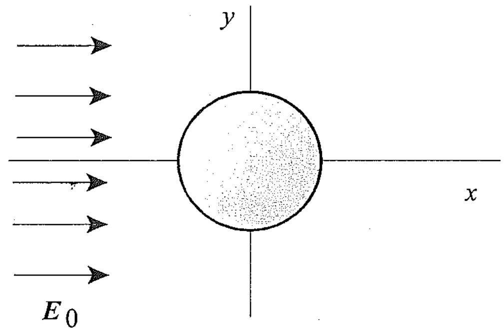
B) Show that the potential difference between a point at $|\vec{r}| = 2c$ and the center of the inner cylinder is:

$$\Delta\Phi = \frac{\lambda_1 + \lambda_3}{2\pi\varepsilon_0}\ln 2 + \frac{1}{2\pi\varepsilon_3}\left[\left(\lambda_1 - \lambda_3\frac{b^2}{(c^2 - b^2)}\right)\ln\frac{c}{b} + \frac{\lambda_3}{2}\right] + \frac{\lambda_1}{2\pi\varepsilon_2}\ln\frac{b}{a}$$



[Hint: Use Gauss' law.]

Ph.D. Comprehensive Examination
Spring 2015
Electricity and magnetism: E&M 600-2



An infinite cylinder of radius a, made of polarizable material with electric susceptibility $\chi > 0$, is immersed in a uniform external electric field $\vec{E}_0 \equiv E_0 \hat{\mathbf{e}}_x$, with the cylinder centered on the z axis.

20 a) Write the appropriate Maxwell equations for $\vec{D}$ and $\vec{E}$, and the appropriate constitutive relations between $\vec{D}$, $\vec{E}$ and $\vec{P}$, assuming an isotropic linear response of the medium.

5 b) Explain why the Laplace equation for the electrostatic potential Φ applies for $r < a$ and $r > a$, but not for $r = a$.

15 c) State the corresponding boundary conditions for $\vec{D}$, $\vec{E}$ and Φ at $r = a$.

15 d) Derive or recall the general solution of the Laplace equation in two-dimensional cylindrical coordinates (r, φ). Apply it to acceptable solutions for $\Phi^{(\mathrm{in})}$ when $r \to 0$ and $\Phi^{(\mathrm{out})}$ when $r \to \infty$.

15 e) Applying the appropriate boundary conditions at $r = a$, determine $\Phi(r,\varphi)$ everywhere, and represent it in Cartesian coordinates as $\Phi(x,y)$.

10 f) In Cartesian coordinates, determine $\vec{E}^{(in)}$, $\vec{D}^{(in)}$ and the polarization vector $\vec{P}$ for $r < a$.

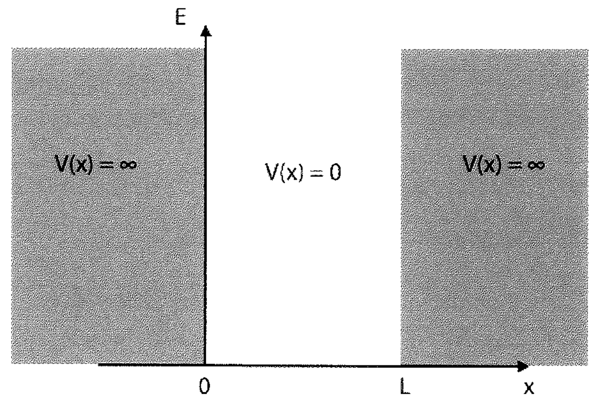
10 g) Determine the surface polarization charge density σ_{pol} at $r=a$.

10 h) In Cartesian coordinates, determine $\vec{E}^{(out)}$ for $r > a$. Sketch the lines of force of $\vec{E}$ in all space.

A particle of mass m is bound in a one-dimensional infinite square well potential. The general Schrödinger equation for is:

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left[\frac{-\hbar^2}{2\mu}\nabla^2 + V(\mathbf{r},t)\right]\Psi(\mathbf{r},t)$$

where μ is the reduced mass.



a) Find $\Psi(x,t)$ that describes the time-dependent motion of the particle.

b) What is the total energy of the particle and what are the eigenvalues?

c) Describe/illustrate where the particle is more likely to be found for eigenvalues 1, 2 and 3.

d) For an eigenstate n of the time-independent Schrödinger equation, the probability density of finding the particle at a given position is $P_n(x) = |\psi_n(x)|^2$. Show that the expectation value of the position of the particle is $L/2$.

e) The variance of the position of the particle is

$$Var(x) = \int (x - \langle x \rangle)^2 P_n(x)dx \ ,$$

and the variance of the momentum is

$$Var(\rho) = \left(\frac{nh}{2L}\right)^2 .$$

Show that the particle obeys the Heisenberg's uncertainty principle.

Consider a spin-3/2 ($s = 3/2$) particle. The normalized eigenfunctions $|s\,m\rangle$ of the operators $\boldsymbol{S}^2$ and S_z can be represented by four-dimensional column matrices

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\ \text{and}\ \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

a) Construct matrices representing the operators $\boldsymbol{S}^2$, S_z, S_+, S_-, S_x, and S_y in this representation.

Hint: The operators $S_\pm$ are defined by the equations

$$S_\pm = S_x \pm i S_y$$
$$S_+|s\,m\rangle = \hbar\sqrt{s(s+1) - m(m+1)}\,|s\,m+1\rangle$$
$$S_-|s\,m\rangle = \hbar\sqrt{s(s+1) - m(m-1)}\,|s\,m-1\rangle$$

b) Of the six matrices constructed in part (a), which are hermitian?

c) An arbitrary spin state in this space can be represented by the column matrix $\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$. Find the expectation value of S_z in this state.

Part (d) does not depend on parts (a)-(c).

d) Consider a particle with spin $s = 3/2$ and orbital angular momentum $l = 2$. What are the possible quantum numbers j, m_j which describe the total angular momentum $\vec{J} = \vec{L} + \vec{S}$?

Ph. D. Comprehensive Exam
Spring 2015
Quantum Mechanics QM 600-2

suggested - AS

Consider a spin-½ particle of mass m and charge q placed in a magnetic field $\vec{B} = (B_1 \cos\omega t, 0, B_0)$. B_0 and B_1 are constants.

(a) Write down an expression for the magnetic moment of the particle, and the potential energy of interaction of the particle with the magnetic field. (20)

(b) Ignoring the particle motion, what is the Hamiltonian for the system? (20)

(c) Find the energy eigenvalues E_+ and E_- and eigenstates $|-\rangle$ and $|+\rangle$ of the particle when $B_1 = 0$. (60)

(d) Treating interaction of the particle with the oscillating magnetic field as a weak perturbation, use first-order time-dependent perturbation theory to find the probability $p(t)$ that, if the system is in the state $|-\rangle$ at $t = 0$, it is found to be in the state $|+\rangle$ at a time $t > 0$.

Linearly polarized electromagnetic waves of the appropriate frequency can excite hydrogen atoms from the state $|nlm\rangle = |210\rangle$ to either of the states $|320\rangle$ or $|300\rangle$. Assume that the wave is polarized along the z axis, and is propagating in the positive x direction.

a) Write down an expression for the perturbation.

b) Write a *general* expression for the transition from state nlm to state $n'l'm'$.

c) Calculate the ratio of the transition probabilities for transitions to the two states given above. Begin by writing down the relevant quantity (quantities) to be calculated and then proceed stepwise.

Some useful relations:

Selected $m = 0$ eigenfunctions for the hydrogen atom:

$$\Psi_{100} = \frac{a_o^{-3/2}}{\sqrt{\pi}} e^{-r/a_o} = 2a_o^{-3/2} e^{-r/a_o} Y_{00}$$

$$\Psi_{210} = \frac{a_o^{-3/2}}{4\sqrt{2\pi}} \frac{r}{a_o} e^{-r/2a_o} cos\theta = \sqrt{\frac{3}{2}} \frac{a_o^{-3/2}}{4} \frac{r}{a_o} e^{-r/2a_o} Y_{10}$$

$$\Psi_{300} = \frac{a_o^{-3/2}}{81\sqrt{3\pi}} (27 - 18\frac{r}{a_o} + 2\frac{r^2}{a_o^2}) e^{-r/3a_o} = \frac{2a_o^{-3/2}}{81\sqrt{3\pi}} (27 - 18\frac{r}{a_o} + 2\frac{r^2}{a_o^2}) e^{-r/3a_o} Y_{00}$$

$$\Psi_{320} = \frac{a_o^{-3/2}}{81\sqrt{6\pi}} \frac{r^2}{a_o^2} e^{-r/a_o} (3cos^2\theta - 1) = \frac{a_o^{-3/2}}{81} \sqrt{\frac{8}{15}} \frac{r^2}{a_o^{3/2}} e^{-r/3a_o} Y_{20}$$

Recursion relation:

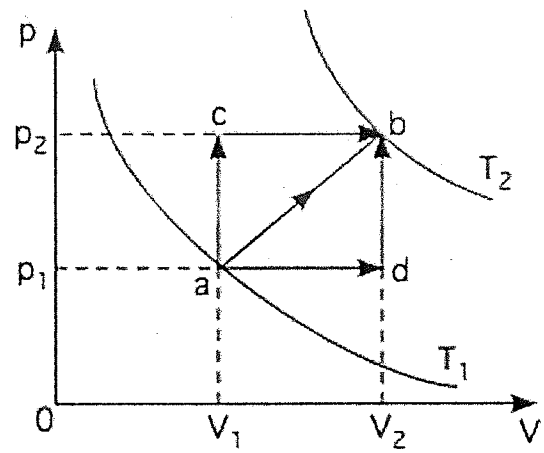
$$cos\theta \, Y_{lm} = [\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}]^{1/2} Y_{l+1,m} - [\frac{(l+m)(l-m)}{(2l-1)(2l+1)}]^{1/2} Y_{l-1,m}$$

A classical ideal gas is taken from state a to state b in the figure using three different paths: acb, adb and ab. The pressure $p_2 = 2p_1$ and the volume $V_2 = 2V_1$.

a) The heat capacity $C_V = 5/2\ Nk$. Starting from the First Law of Thermodynamics derive a value for C_p. No credit will be given for stating the answer.

b) Compute the heat supplied to the gas along each of the three paths *acb*, *adb* and *ab* in terms of N, k and T_1.

c) What is the heat capacity C_{ab} of the gas for the process *ab*?



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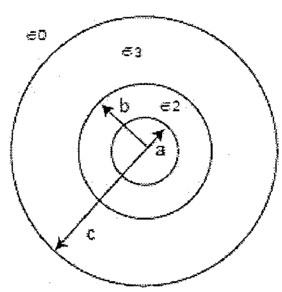

[Hint: Use Gauss' law.]

(a) Noting that for the grand canonical ensemble

$\langle S \rangle = -k_B \sum_{i,N} p_{i,N} \ln p_{i,N}$ and $p_{i,N} = \dfrac{e^{-\beta(E_i - \mu N)}}{Z_G}$, show that

$\langle S \rangle = k_B [\beta \langle E \rangle - \mu \beta \langle N \rangle + \ln Z_G]$.

Here, the index i labels the states of the system and N stands for the number of particles;

$\beta = 1/k_B T$ and μ is the chemical potential.

(b) Using $\Omega(T, V, \mu) = U - TS - \mu N$, show that $\Omega(T, V, \mu) = -PV$. Here, U is the internal energy.

(Hint: By observing that the grand potential $\Omega(T, V, \mu)$ is a first order homogenous function of V only, you should be able to derive the result using Euler's theorem on homogenous functions.)

(c) Now, using your results from parts *a* and *b*, show that $\Omega(T, V, \mu) = -k_B T \ln Z_G$.

(d) Derive or write down expressions for $< E >$ and $ln\ Z_G$ for non-interacting quantum gases of Bosons and Fermions in three dimensions. Explain your notation.

(e) Using integration by parts, show that $PV = \gamma < E >$ for a non-interacting quantum gas of Fermions with $\varepsilon_p = p^2/(2m)$ in three dimensions, and find the dimensionless proportionality constant γ. This result does not depend on the particle degeneracy M.

(Note that from part *c*, $\beta PV = ln\ Z_G$.)