Two masses m_1 and m_2 are joined by a massless spring (force constant k and natural length l_0) and are confined to move in a frictionless horizontal plane, with center of mass (CM) position $\vec{R}$ and relative position $\vec{r}$.

a) What are the generalized coordinates of the system?

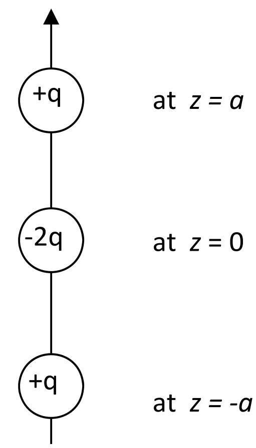
b) Write down the Hamiltonian using the generalized coordinates.

c) Which are the ignorable (cyclic) coordinates? Explain.

d) Write down all the Hamilton equations of motion.

e) Combine the equations for r and write down a single radial equation. Solve the radial equation for the special case where one of the conjugate momenta is zero.

Consider three point charges along the z axis as shown in the diagram.



a) Calculate the potential at a point on the positive z axis for $z \gg a$.

b) Expand the result from part (a) in powers of z to show that the lowest non-vanishing multipole moment is a quadrupole.

c) Imagine that the three point charges shown are all connected by insulating rods to make a rigid structure. If placed in a uniform static electric field, would this configuration of point charges experience a net force? A net torque? Explain your reasoning.

d) Draw a non-collinear [*i.e.,* not all lying along one line] configuration of point charges that also has a quadrupole moment as its lowest non-vanishing multipole moment.

A particle is in a potential of the form $V(x) = 0$ for $0 < x < a$, and $V(x) = \infty$ otherwise.

(a) Find the eigenfunctions $\psi_n(x)$ and energy eigenvalues E_n.

(b) Calculate $\Delta x = \sqrt{\left\langle \left(x - \langle x \rangle\right)^2 \right\rangle}$ when the system is in the state $\psi_n(x)$.

(c) Calculate $\Delta p = \sqrt{\left\langle \left(x - \langle x \rangle\right)^2 \right\rangle}$, where p is the momentum, when the system is in the state $\psi_n(x)$.

(d) Calculate $\Delta p \Delta x$ when the system is in the state $\psi_n(x)$ and show that the result is consistent with the uncertainty principle for all n.

Suppose now that the system is in the state $\psi(x) = A\left(\sin\frac{\pi x}{a}\right)^5$.

(e) Find the corresponding time dependent wave functions $\Psi(x,t)$.

[Hint: You can avoid doing any integrals by first writing the *sin* function in terms of complex exponentials.]

(f) Find the normalization factor A.

(g) Find the probability that a measurement of the energy yields the value E_3.

In a refrigerator, we are interested in the temperature change produced by a gas expansion characterized by the rate of change of temperature with pressure at constant enthalpy (the Joule-Thomson coefficient).

a) Derive the following expression for the Joule-Thomson coefficient, μ:

$$\mu = \frac{V}{C_P}(T\alpha - 1),$$

where V is the volume, C_P is the heat capacity at constant pressure, T is the absolute temperature, and α is the coefficient of thermal expansion.

b) Show that the Joule-Thomson coefficient of an ideal gas is zero.

c) Find the condition for the Joule-Thomson coefficient in an imperfect gas to be positive, so that a drop in pressure produces a drop in temperature.

Consider now an N-particle imperfect gas that obeys the Equation of State

$$P = \frac{N}{V}kT(1 + \frac{N}{V}B),$$

where B, which characterizes the non-ideality, is a parameter that is negative at low temperature and increases with temperature, becoming positive at high temperatures.

d) Provide an explanation for the temperature dependence of B in terms of inter-particle interactions.

e) Show that the Joule-Thomson coefficient for this gas changes sign at some temperature (the “inversion temperature”) and provide a qualitative explanation for this behavior.

A long pendulum is suspended from a point (0,0,l) above ground on the northern hemisphere and brought to swing in a vertical plane with small amplitude.

(a) Let $(\hat{x},\hat{y},\hat{z})$ be the axes of a coordinate system associated to the earth with $\hat{z}$ directed upwards and $\hat{x}$ along a geodesics facing towards the equator. Obtain the equation of motion for this system.
Note: The tension on the thread is $\vec{F}_{tens} = |F_{tens}| \left(\left(1-\frac{z}{l}\right)\hat{z} - \frac{x}{l}\hat{x} - \frac{y}{l}\hat{y}\right)$

(b) Ignoring the centrifugal force and any velocity pointing outwards (to the sky), i.e., $0 \approx \dot{z} << \dot{x},\dot{y}$, solve the equation of motion and find the frequency of horizontal angular deflection ($\alpha = {}^{d\varphi}\!/_{dt}$) of the pendulum.

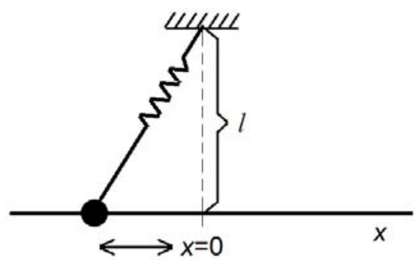
(c) Determine the horizontal projection of the path described by the pendulum and its velocity for the start conditions: $x(0) = a$; $y(0) = 0$; $\dot{x}(0) = \dot{y}(0) = 0$.
Determine the times and velocities when the pendulum arrives at turning points. Sketch the projection of the pendulum's path on the horizontal plane.

(d) Describe the motion of the pendulum if it is brought to swing by pushing the bob from the equilibrium point in the y-direction by an initial velocity $v_{0y} = a\sqrt{{}^{g}\!/_{l}} = a\,\omega_0$.
Determine the times and velocities when the pendulum arrives at turning points. Sketch the projection of the pendulum's path on the horizontal plane.

Consider a mass m which is constrained to move on a straight line. The mass is bound to a fixed point by harmonic force with potential energy

$$V = \frac{1}{2}K(r-R)^2,$$

where K is a constant, R is the rest length of the spring, and r is the distance of the particle from the fixed point. The distance from the point to the line is $l > R$. A mechanical model of this system is a mass sliding on a straight track; the mass being connected to a fixed point by a spring, as shown in the figure.



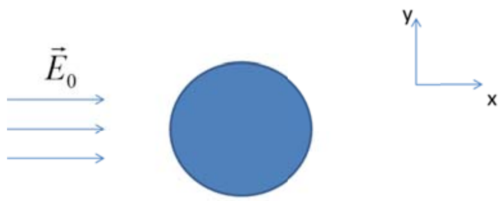
(a) Obtain the Lagrangian of this system.

(b) Find the stable equilibrium position(s) of the system.

(c) Obtain a simplified Lagrangian describing the harmonic approximation near equilibrium. *Hint:* Expand the Lagrangian in a Taylor series in x keeping only the first nontrivial terms.

(d) Using the obtained Lagrangian, compute the frequency of the small oscillations about the equilibrium.

An infinite cylinder of radius *a*, made up of electrically polarizable material, is immersed in an external uniform electric field.



$\vec{P} = \beta \varepsilon_0 \hat{z} \times \vec{E}$

The material has unique polarizability properties. In the presence of an electric field, the dipoles tend to line up transverse to the external field, so that

$$\vec{P} = \beta \varepsilon_0 \hat{z} \times \vec{E} ,$$

where β is a constant and $\hat{z}$ is the unit vector in the *z* direction.

(a) Find the electric potential inside and outside the cylinder.

(b) Examine the two limits $\beta \to 0$ and $\beta \to \infty$. What are the potentials in these limits.

(c) Switch to Cartesian coordinates for the solution inside the cylinder and calculate the electric field using Cartesian unit vectors. Sketch the electric field lines inside and outside for arbitrary and the limiting values of β .

Consider a surface current density on a cylindrical surface of radius a, given by the expression:

$$\vec{K} = K_0 \cos(\emptyset)\, \hat{z}$$

The cylindrical surface is infinite in the *z*-direction.

(a) Find the vector potential inside and outside the cylindrical surface. Do not use the Biot-Savart law.

(b) Find the auxiliary field *H* inside and outside the cylindrical surface from the vector potential.

(c) Sketch the magnetic field inside and outside the cylinder.

For the questions below, assume that the cylinder is solid and that it is uniformly magnetized along the *z*-direction and where, unlike in parts (a)-(c), there are no additional surface currents.

(d) Find the vector potential inside and outside the cylinder.

(e) Check whether the vector potentials in (d) satisfy the Coulomb gauge condition.

(f) Check whether the vector potentials in (d) satisfy the boundary conditions.

Consider a rest frame K′ in which a source point particle with charge q_s stays fixed at the origin. In K′, a mobile test point particle with charge q_{ts} is initially placed at $t' = 0$ at a position $\vec{y}_0' = y_0'\hat{e}_y'$ on the y′-axis at a distance y_0' from q_s. Initially q_{ts} is also at rest.

(a) Find the electric field $\vec{E}'(0)$, the force $\vec{F}'(0)$, and the acceleration $\vec{a}'(0)$ acting on q_{ts} as a result of q_s at the initial time.

We now wish to study the same physical problem in a (laboratory) frame K relative to which K′ slides with constant velocity $\vec{w} = w\hat{e}_x'$ along their common x- and x'- axes.

(b) Make a sketch of the two relatively moving inertial frames K and K′ and mark the position of q_s.

(c) Write the Lorentz transformation among the coordinates (ct, x, y, z) and (ct', x', y', z') in K and K′, respectively.

An electromagnetic field appears in K, related to that in K′, as $\vec{E}_\parallel = \vec{E}_\parallel'$, $\vec{E}_\perp = \gamma \vec{E}_\perp'$, $\vec{B}_\parallel = \vec{B}_\parallel' = 0$, $\vec{B}_\perp = \gamma \frac{1}{c} \vec{w} \times \vec{E}'$.

(d) By considering the non-relativistic limit of $w \ll c$ and $\gamma \simeq 1$, provide an intuitive justification for the appearance of $\vec{B}$ in K.

(e) Determine the Lorentz force acting on the test point particle q_{ts} in K initially at $t = t' = 0$ (at $x = x' = 0$).

(f) Determine the equation of motion for q_{ts} initially.

(g) Determine the accelerations $\vec{a}(0)$ and $\vec{a}'(0)$ of q_{ts} in K and K′ and comment on their relative magnitudes, especially in the non-relativistic and ultra-relativistic limits.

An electron of charge *e* moves in one dimension and is confined to the right half space (*x>0*). The plane *x=0* is an infinite perfect conductor so that the effective potential for the electron is due to its image charge.

a) What is that potential?

b) What boundary condition(s) must the electron's wave function satisfy?

c) What is the ground state energy of the electron?

d) How far, on average, is the electron from the conductor?

The following integral may be useful:

$$\int_0^\infty x^n \, e^{-2ax} dx = \frac{\Gamma(n+1)}{a^{n+1}}, \qquad \Gamma(n+1) = n\,!$$

Consider a particle of mass μ and charge q constrained to move in one dimension, which is subjected to a harmonic oscillator potential of frequency ω.

a) Define appropriate operators $\boldsymbol{a}, \boldsymbol{a}^{+}$ as combinations of the $\boldsymbol{X}$ and $\boldsymbol{P}$ operators to factorize the Hamiltonian $\boldsymbol{H}$.

b) What are the energy eigenkets and eigenvalues?

Now consider this charged oscillator in a homogeneous electrostatic field in positive x direction.

c) By completing the square find the translation that allows the new Hamiltonian H' to be written as a harmonic oscillator up to a constant. What are the energy eigenvalues?

d) Show that the action of the translation operator $\boldsymbol{T}(l) = e^{-\frac{i}{\hbar}\boldsymbol{P}l}$ is described by the transformation $\boldsymbol{T}^{+}(l)\, \boldsymbol{X}\, \boldsymbol{T}(l) = \boldsymbol{X} + l$ and write the new energy eigenkets as $|\bar{n}\rangle = \boldsymbol{T}(l)|n\rangle$ using an appropriate constant l.

e) If the particle is initially in the ground state of the original Hamiltonian $\boldsymbol{H}$ from part (a), what is the probability of finding it in the ground state of the new Hamiltonian $\boldsymbol{H}'$ from part (c)?

In first-order time-dependent perturbation theory, the transition probability amplitude from an initial state $|i\rangle$ at time t_i to a final state $|f\rangle$ at time t_f is given by the coefficient

$$a_{i\to f} = \frac{1}{i\hbar}\int_{t_i}^{t_f} dt \; e^{i(E_f - E_i)/\hbar} \langle f|H_1(t)|i\rangle$$

where $H_1(t)$ is the perturbing Hamiltonian. Consider a harmonic perturbation of the form

$$H_1(t) = H' e^{-i\omega t} + H'^{\dagger} e^{i\omega t}$$

acting only in the time interval T between $t_i = 0$ and $t_f = T > 0$. Assume that $\omega_{fi} = \frac{1}{\hbar}\left(E_f - E_i\right) > 0$.

(a) Determine the corresponding probability amplitude $a_{i\to f}(T)$.

(b) Consider an absorption process, for which $\omega \sim \omega_{fi} > 0$. Determine the corresponding transition probability $P_{i\to f}(T) = \left|a_{i\to f}(T)\right|^2$.

(c) Draw a plot of $\left\{\frac{\sin\left[\frac{1}{2}(\omega-\omega_{fi})T\right]}{\left[\frac{1}{2}(\omega-\omega_{fi})\right]}\right\}^2$ as a function of $(\omega - \omega_{fi})$ for given T.

Argue that for $T \to +\infty$, that function approaches $2\pi T\delta(\omega - \omega_{fi})$. An explicit calculation of the coefficient is not required, however.

(d) From these considerations, obtain Fermi's Golden Rule for the Transition Probability Rate, i.e., per unit time.

An N-particle monatomic ideal gas is contained in a vertical cylinder of area A that is sealed with a piston of mass M that is free to move without friction. In this problem, the effect of gravity on the piston must be included but its effect on the particles of the gas can be neglected.

(a) Describing the state of the piston by a single coordinate z_0 and momentum p_0, write out the Hamiltonian for the system.

(b) Calculate the canonical partition function.

(c) Using the canonical ensemble, calculate the probability distribution $P(z_0)$ for the piston height.

(d) Determine the most probable height, z^*, and the average height, $\langle z_0 \rangle$, of the piston at a given temperature T for the gas in the cylinder.

(e) Show that for large N, $P(z_0)$ is a Gaussian function.

(f) Show that the fluctuation in the gas volume is given by $\dfrac{\Delta V}{V} = \dfrac{1}{\sqrt{N}}$.

The following results may be useful:

$$N! = \int_0^\infty x^N e^{-x} dx$$

$$N! \approx N^N e^{-N} (2\pi N)^{1/2} \quad \text{for large } N.$$

A DNA molecule can be crudely modeled as a pair of parallel strands, each composed of N sequential units. In the ground state configuration, each pair of corresponding units on the two strands are chemically linked. In order to break one link, an energy $\varepsilon > 0$ must be provided to the pair of units facing each other. It is also required that either the pair sits at one or the other end of the double strand, or, if not, that at least one of the two links adjacent to the link that could be opened with energy ε is already open. In other words, the double strand can only be unzipped sequentially. It would take infinite energy to open a link if both of its adjacent links are still closed. An example of a possible unzipping configuration is sketched in Figure 1.

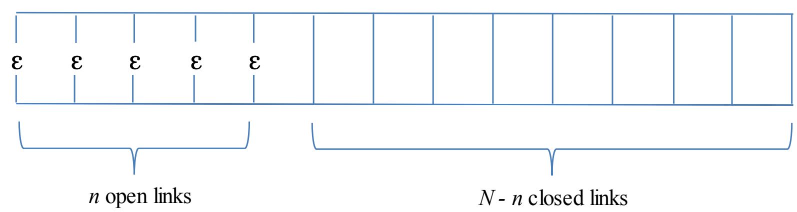


Figure 1

(a) For now, let us also suppose that the N-th link at the right end of the double strand is externally blocked from opening. In other words, the double strand can only unzip starting from the left end. Determine the Canonical Partition Function $Q_N(\beta)$ for a single double strand of N units at an absolute temperature such that $\beta = 1/k_B T$. You may set for convenience $x \equiv e^{-\beta\varepsilon}$ and recall that $\sum_{n=0}^{N-1} x^n = \frac{1-x^N}{1-x}$ for $0 < x < 1$.

(b) Determine the Canonical Probability $P_n(\beta)$ that n links are opened from the left end of the double strand of N units at temperature T.

(c) Determine the average number $\langle n \rangle = \sum_{n=0}^{N-1} nP_n$ of open links at T. How is this result related to the average excitation energy $U = -\frac{\partial}{\partial\beta}[\log Q_N(\beta)]$?

Let us now make the alternative hypothesis that the double strand can be equally opened from both ends. In Figure 2, a possible configuration is sketched, in which n_1 links are opened from the left end, while n_2 links are opened from the right end, generating a total of $n = n_1 + n_2$ open links.

{problem continues on the next page}

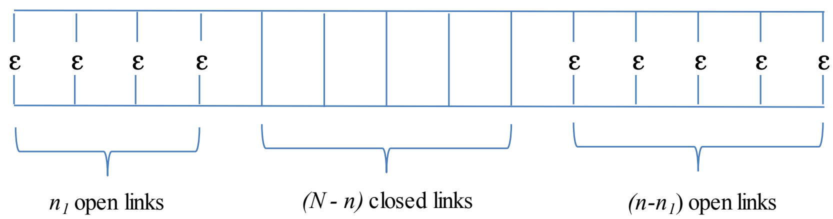


Figure 2

(d) Determine the degeneracy g_n of the states that have a total of n open links from both ends. Distinguish the case of a totally unzipped double strand, for which $n = N$, from the cases of a partially unzipped strand, for which $n \leq N - 1$. The $n = 0$ case of a totally closed double strand is a particular instance of those with $n < N$.

(e) Determine the Canonical Partition Function $Z_n(\beta)$ for a single double strand that can be unzipped from both ends at temperature T.

(f) Determine the Canonical Probability $p_n(\beta)$ that a total of n links can be open from both ends of a double strand at temperature T.