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THE CATHOLIC UNIVERSITY OF AMERICA

*Department of Physics*
*200 Hannan Hall*
*Washington, D.C. 20064*
*202-319-5315*
*Fax 202-319-4448*

**Ph.D. Comprehensive Examination**

**Physics Department**

**SPRING 2014**

**Thursday, March 27 and Friday, March 28, 2014**

**GENERAL INSTRUCTIONS:**

This examination is divided into four sections as follows:

| | | |
|---|---|---|
| Thursday. March 27, 2014 | | |
| 9:00 am 12:00 Noon | Classical Mechanics | - 2 questions |
| 1:00 pm 5:00 pm | Electricity & Magnetism | - 3 questions |
| | | |
| Friday. March 28. 2014 | | |
| 9:00 am 12:00 Noon | Statistical Mechanics | - 2 questions |
| 1:00 pm 5:00 pm | Quantum Mechanics | - 3 questions |

In each of the four subject areas, you may answer the **500-level question** in place of a 600-level question.

**DO EACH PROBLEM IN A SEPARATE BLUE BOOK**

**PUT YOUR NAME ON EACH BOOK**

**LABEL EACH BOOK WITH THE CORRECT PROBLEM NUMBER, FOR EXAMPLE: "Mechanics 600-1"**

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*http://physics. cua. edu*

# RULES FOR THE PHYSICS COMPREHENSIVE EXAMINATION

## Spring 2014 -- March 27 & 28, 2014

In order to assure an equitable and fair Comprehensive Examination for all students, the following rules will apply to the examination administered by the Department of Physics.

1. The examination will be **closed book**. Students will not be permitted to bring to the examination room materials such as notes, reference books, calculators, or other aids of any form. The Physics Department will supply copies of the *CRC Mathematical Handbook,* Schaum's *Mathematical Handbook,* the *Table of Functions* by Jahnke and Emde, and the *NBS Handbook of Mathematical Functions,* for use during the examination.

   Students who want to use their own mathematical tables and/or handbooks must get permission from the Chairman prior to the beginning of the comprehensive examination. These tables and/or handbooks will remain under the control of the Department until after the examination is completed, but will be available to their owners during the examination period.

   The Physics Department will supply calculators for use during the examination.

2. If needed, students will be allowed a short break, not to exceed ten minutes, during the examination period. While absent from the examination room students may not consult material or other sources of information concerning any matter pertaining to the examination. Students who leave the examination room will turn in all of their examination material to the proctor before leaving the room. Examination papers will be returned to the students when they reenter the examination room. Students who fail to return within ten minutes will be considered to have completed that portion of the examination.

   Only one student will be permitted to leave the examination room at a time.

3. Students are not permitted to bring any electronic devices (including cell phones, PDAs and calculators) into the examination room. Any such devices must be handed over to the proctor for the duration of the exam session.

Consider a spin-3/2 ($s = 3/2$) particle. (Such particles are known in particle physics since the 1960's.) Such a spin can have four possible projections along the z axis, and the normalized eigenfunctions $|s\,m\rangle$ of the operators $\boldsymbol{S}^2$ and $S_z$ can be represented by four-dimensional column matrices

$$\left|\tfrac{3}{2}\;\tfrac{3}{2}\right\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \left|\tfrac{3}{2}\;\tfrac{1}{2}\right\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad \left|\tfrac{3}{2}\;-\tfrac{1}{2}\right\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad \text{and} \quad \left|\tfrac{3}{2}\;-\tfrac{3}{2}\right\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

a) Construct matrices representing the operators $\boldsymbol{S}^2$, $S_z$, $S_+$, $S_-$, $S_x$, and $S_y$ in this representation.

Hint: The operators $S_\pm$ are defined by the equations

$$S_\pm = S_x \pm iS_y$$
$$S_+|s\,m\rangle = \hbar\sqrt{s(s+1) - m(m+1)}\,|s\,m+1\rangle$$
$$S_-|s\,m\rangle = \hbar\sqrt{s(s+1) - m(m-1)}\,|s\,m-1\rangle$$

b) Of the six matrices constructed in part (a), which ones are hermitian?

c) An arbitrary spin state in this space can be represented by the column matrix $\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$, where $a$, $b$, $c$ and $d$ are complex numbers. Find the expectation value of $S_z$ in this state.

Parts (d) and (e) do not depend on parts (a)-(c).

d) Consider a particle with spin $s = 3/2$ and orbital angular momentum $l = 1$. What are the possible quantum numbers $j$, $m_j$ which describe the total angular momentum $\vec{J} = \vec{L} + \vec{S}$?

e) Find the total number of states in the $|j\,m_j\rangle$ representation, and show that it is the same as the total number of states in the $|l\,m_l\,s\,m_s\rangle$ representation.

A one-dimensional harmonic oscillator (H.O.) has its equilibrium position displaced over time, so that its potential energy is

$$V(x) = \frac{1}{2}m\omega^2\left[x - x_0(t)\right]^2 = \frac{1}{2}m\omega^2 x^2 + H'(t),$$

where $H'(t)$ may be regarded as a time-dependent perturbation Hamiltonian. Specifically, consider

$$x_0(t) = x_M e^{-t^2/(2\tau^2)}$$

as the origin displacement of the spring force.

Suppose that at the initial time $t_i = -\infty$, the H.O. is in its ground state $|i\rangle = |0\rangle$, while at final time $t_f = +\infty$ the perturbation has induced the H.O. to acquire a certain admixture of the first excited state $|f\rangle = |1\rangle$.

a) Show that the corresponding probability of excitation is

$$P_{i\to f} = \pi\left(\frac{m\omega}{\hbar}\right)(\omega\tau)^2 e^{-(\omega\tau)^2} \cdot x_M^2$$

In performing your derivation, you may use the following formulas, but you must explain briefly their meaning and justify their use:

$$(1)\quad a_{i\to f} = \frac{1}{i\hbar}\int_{t_i}^{t_f} dt\, e^{\frac{i}{\hbar}(E_f - E_i)t}\langle f|H'(t)|i\rangle \qquad (2)\quad \langle 1|x|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}$$

$$(3)\quad \int_{-\infty}^{+\infty} dt\, e^{i\omega t - \frac{t^2}{2\tau^2}} = \sqrt{2\pi\tau^2}\, e^{-\frac{1}{2}(\omega\tau)^2}$$

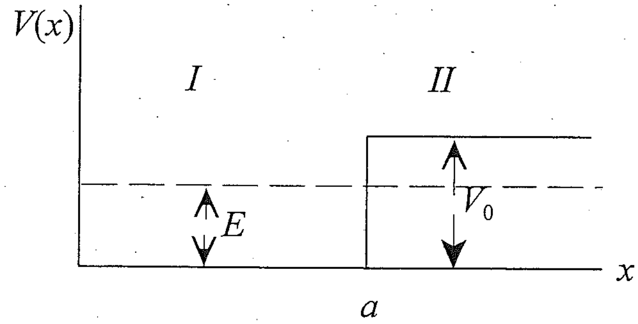
b) Plot $P_{i\to f}$ as a function of $\tau$, and find its maximum value and location. Interpret physically why the probability of excitation is greatest at that particular value of $\tau$.

c) What condition is required of $P_{i\to f}$ for first-order time-dependent perturbation theory to apply? What limits for the relation between $\tau$ and $\omega$ provide this condition? What are the physical interpretations of these two limits?

d) In the ground state, a measure of a H.O. amplitude is given by $(\Delta x)_0 = \sqrt{\dfrac{\hbar}{2m\omega^2}}$.
Suppose that $\omega\tau = 1$. What condition on the maximum origin displacement $x_M$ makes your perturbative result valid? What is the physical interpretation of that condition?

A particle of mass $m$ is bound in a semi-infinite potential well

$$V(x) = \begin{cases} \infty & x \le 0 \\ 0 & 0<x<a \quad \text{(Region I)} \\ V_0 & x \ge a \quad \text{(Region II)} \end{cases}$$




Assume that the particle has energy $0 < E < V_0$.

a) Write the general form of the solutions to the time-independent Schrödinger equation in Region I and in Region II. Call these solutions $\psi_{I}(x)$ and $\psi_{II}(x)$.

b) Applying the appropriate boundary conditions at 0 and $\infty$, simplify the general forms you wrote in part (a). Explain your reasoning.

c) Applying the boundary conditions at $x=a$, write the simultaneous equations which you could solve to find (i) the relationship between the constants in $\psi_{I}(x)$ and $\psi_{II}(x)$ and (ii) the energy $E$. (Do not solve.)

The following parts are independent of (a)-(c):

Assume that the two lowest-energy solutions of the time-independent Schrödinger equation for this potential are

$\psi_1(x)$ with energy $E_1$ and $\psi_2(x)$ with energy $E_2$.

[$\psi_1(x)$ and $\psi_2(x)$ both have different forms in Region I and Region II, but that is of no concern here.] Assume that $\psi_1(x)$ and $\psi_2(x)$ are both real.

d) Write the general solution $\Psi(x, t)$ of the time-dependent Schrödinger equation in terms of $\psi_1(x)$ and $\psi_2(x)$.

e) Show that the probability distribution $|\Psi(x,t)|^2$ has a time-independent part and a time-dependent part, and find the time dependence.

A particle of mass M is in a stable circular orbit around a string with a angular momentum L. Consider this string to be infinitely long and with a linear mass density $\lambda$.

a) Find the force on M produced by the string

b) Considering the centrifugal contribution, find the effective potential, $V_{eff}(r)$, for the particle

c) What is the radius of the circular orbit?

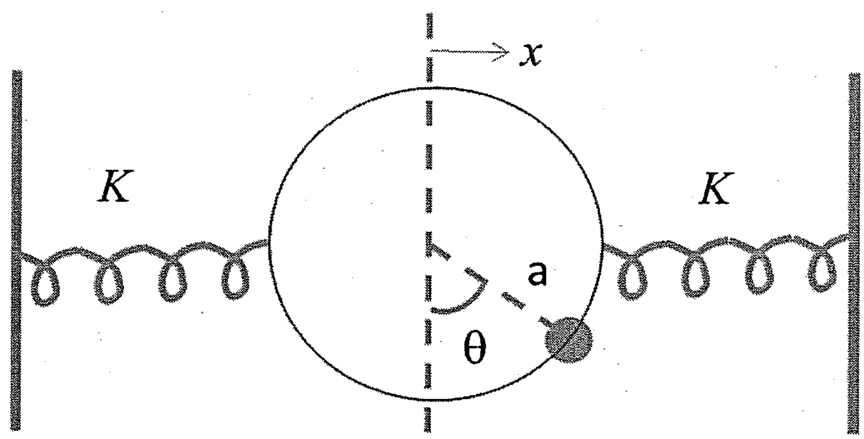
d) What is the frequency of small radial oscillations around the circular orbit?

A particle of mass $m$ can slide without friction along a massless circular wire frame of radius $a$. The frame is supported on either side by identical massless springs of spring constant $K$. The springs can only extend in the $x$-direction; they are perfectly rigid in the other two directions. In the figure, $x$ is the displacement of the center of wire frame.

a) Find the Lagrangian for this system and the associated equations of motion.

b) Find the Hamiltonian for the system and the generalized momenta for the system.

c) Find a constant of motion for the system and explain why it is conserved.



Ph. D. Comprehensive Exam, Spring 2014
Classical Mechanics CM 600-2

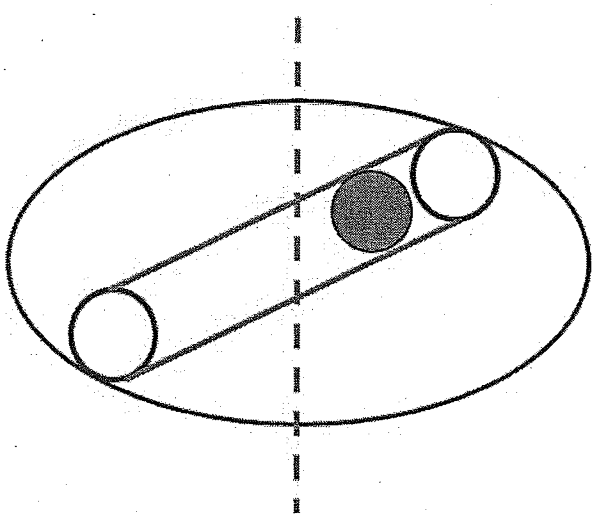
A hollow massless tube is fixed at the center and can freely rotate in a circle in the plane about a vertical axis passing through the center – see figure. A particle of mass $m$ is moving without friction inside the tube. You can take the particle to have the same cross-sectional diameter as the tube; however, for the purpose of finding the Lagrangian, you may take the particle to be a point mass.

a) How many generalized coordinates are needed to describe the system and what are they (the system here is the particle)?

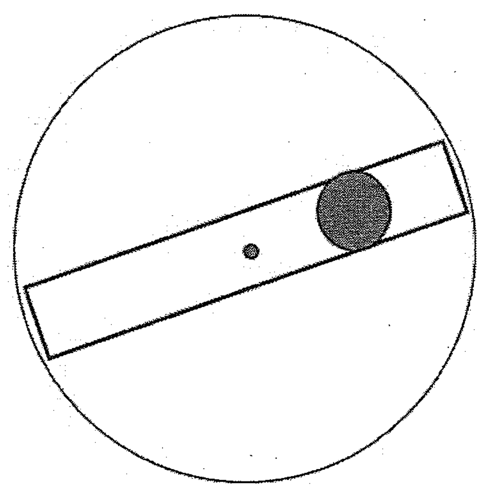
b) Find the Lagrangian for the system and the associated equations of motion.

c) Same set up as above but now the tube is made to rotate at a constant rate $\Omega$. Find the Lagrangian for the system and the associated equation of motion.

d) For the system in part b, find an ignorable coordinate and the associated constant of motion. What is its physical interpretation?



Side view



Top view

Ph. D. Comprehensive Exam, Spring 2014
Statistical Mechanics SM 600-1

Consider a classical one-dimensional simple harmonic oscillator for which the energy is

$$E(x,p) = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

where $-\infty \le p \le +\infty$ & $-\infty \le x \le +\infty$.

Thus, each state of the oscillator is specified by $(x,p)$ where $x$ is the position of the oscillator and $p$ is its momentum.

(a) Show that the partition function for the oscillator is $Z(T) = \left(\frac{K_B T}{\hbar}\right)\sqrt{\frac{m}{k}}$ where $\hbar = \frac{h}{2\pi}$.
(You'll have to do a couple of Gaussian integrals.)

(b) Starting from the properly normalized Boltzmann distribution for the oscillator, integrate over the velocities to obtain the probability distribution for $x$. (Recall that the Boltzmann distribution gives the probability for the oscillator to be between position $x$ and $x+dx$ with momentum $p$ and $p+dp$.)

(c) At temperature $T$, estimate the typical value of $x^2$ excited thermally, i.e. using the result from part c, calculate $<x^2>$.

(d) Some commonly used force transducers are 'cantilevers', or few-micron sized pieces of metal or glass, which bend in response to applied force. For cantilevers with $k = 1$ nN/Å, and $k = 1$ pN/μm, estimate the typical value of $|x|$ generated by thermal fluctuations at room temperature ($T = 300$ $K$.) Note that at room temperature, $1K_BT = 4.1 \times 10^{-21}$ J.

PhD Comprehensive Examination, Spring 2014
Thermodynamics 500-1

**a)** Starting with $U = Q + W$ (internal energy); $H = U + PV$ (Enthalpy); $A = U - TS$ (Helmholtz function) and $G = H - TS$ (Gibb's function) for a closed system, prove the following (Maxwell) relations:

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V; \quad \left(\frac{\partial T}{\partial P}\right)_S = \left(\frac{\partial V}{\partial S}\right)_P; \quad \left(\frac{\partial P}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T; \quad \left(\frac{\partial V}{\partial T}\right)_P = -\left(\frac{\partial S}{\partial P}\right)_T$$

Also prove:

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P$$

**b)** Calculate $\left(\frac{\partial U}{\partial V}\right)_T$ for a van der Waals gas, where

$$P = \frac{RT}{V - B} - \frac{a}{V^2}$$

**c)** Show that your result is compatible with the other equation of state of a van der Waals gas: $U = \frac{3}{2}RT - \frac{a}{V^{\cancel{2}}}$

a) Determine the probability $P_{\text{odd}}$ that a one-dimensional harmonic oscillator of frequency $\omega$ at temperature $T$ is in a quantum state of odd quantum number $n = 1, 3, 5, \ldots$ .

b) Similarly, determine $P_{\text{even}}$ that $n = 0, 2, 4, 6, \ldots$, and verify that $P_{\text{odd}} + P_{\text{even}} = 1$.

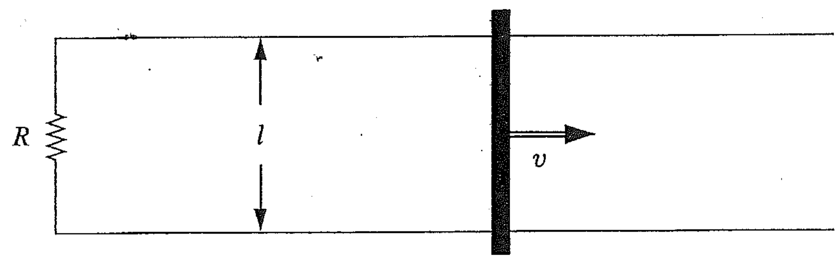
c) Find the limits of $P_{\text{odd}}$ and $P_{\text{even}}$ for $k_BT \ll \hbar\omega$ and explain the physical meaning of their behavior.

d) Find and discuss the corresponding limits of $P_{\text{odd}}$ and $P_{\text{even}}$ for $k_BT \gg \hbar\omega$.

e) What is the $P_{\text{odd}} / P_{\text{even}}$ ratio in general and in the low-temperature and high-temperature limits?

Hint: you may wish to define the parameter $x = \exp(-\hbar\omega / k_BT)$.

A metal bar of mass $m$ slides frictionlessly on two parallel conducting rails, a distance $l$ apart as shown in Fig. 1. A resistor $R$ is connected across the rails and a uniform magnetic field, **B**, pointing into the page, fills the entire region.




(Fig. 1)

(a) If the bar moves to the right at speed $v$, what is the current in the resistor? In what direction does it flow?

(b) What is the magnetic force on the bar? In what direction?

(c) If the bar starts out with speed $v_0$ at time t = 0, and is left to slide, what is its speed at a later time $t$?

(d) The initial kinetic energy of the bar was $\frac{1}{2}mv_0^2$. Check that the energy delivered to the resistor is exactly $\frac{1}{2}mv_0^2$.

PhD Comprehensive Examination, Spring 2014
Electricity and Magnetism 600-1

Using the method of images consider the problem of a point charge $q$ inside a hollow, grounded, conducting sphere of inner radius $a$. Assume that the thickness of the sphere is negligible. Find the following:

a) the potential inside the sphere;

b) the induced surface charge density;

c) the magnitude and direction of the force acting on $q$.

d) How does the solution change if the sphere is kept at a fixed potential V?

e) How does the solution change if the sphere has a total charge $Q$ on its inner and outer surfaces?

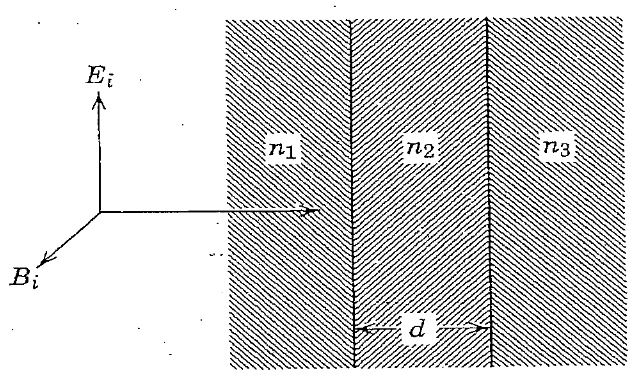
PhD Comprehensive Examination, Spring 2014
Electricity and Magnetism 600-2

A plane wave is incident on a layered interface as shown in the figure below. The indices of refraction of the three non-permeable media are $n_1$, $n_2$, $n_2$. The thickness of the intermediate layer is $d$. The other media are each semi-infinite.

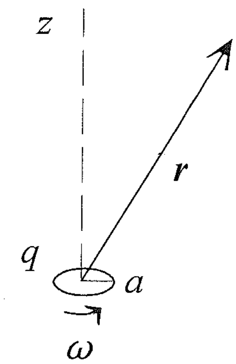
a) Calculate the transmission and reflection coefficients (ratios of transmitted and reflected Poynting's flux to the incident flux), and sketch their behavior as a function of frequency for the following three cases:

i) $n_1 = 1$, $n_2 = 2$, $n_3 = 3$;
ii) $n_1 = 3$, $n_2 = 2$, $n_3 = 1$;
iii) $n_1 = 2$, $n_2 = 4$, $n_3 = 1$.

b) The medium $n_1$ is part of an optical system (e.g., a lens); medium $n_3$ is air ($n_3 = 1$). It is desired to put an optical coating (medium $n_2$) on the surface so that there is no reflected wave for a frequency $\omega_0$. What thickness $d$ and index of refraction $n_2$ are necessary?



A thin charged uniform ring of radius $a$ and total charge $q$ lies in the $z = 0$ plane, centered on the origin, and rotates about the $z$-axis at constant angular velocity $\omega$.




a) Write an exact integral expression for the vector potential $\vec{A}(\vec{r})$ at arbitrary position $\vec{r}$. Define any additional symbols you use. (You need not convert every symbol into a form suitable for integration.)

In the limit $r >> a$, this can be reduced (with much effort) to the result

$$\vec{A}(\vec{r}) = \vec{A}_{dipole}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

where $\vec{m}$ is the magnetic dipole moment of the rotating ring, and $\hat{r} \equiv \dfrac{\vec{r}}{r}$.

b) Find the magnetic dipole moment $\vec{m}$ (magnitude and direction) of the rotating ring.

c) Calculate the magnetic field vector $\vec{B}(\vec{r})$ and the electric field vector $\vec{E}(\vec{r})$ at $\vec{r}$, assuming that $r >> a$. (The direction is an essential part of each answer. Use spherical coordinates.) You may leave your answer in terms of $m \equiv |\vec{m}|$.

d) Calculate the Poynting vector $\vec{S}(\vec{r})$ at $\vec{r}$.

e) Consider a spherical surface of radius $R >> a$ centered on the origin. Calculate the Poynting vector at any position $\theta,\phi$ on the surface, and calculate the flux of $\vec{S}$ through the spherical surface. Is this result expected on general principles? Explain your answer.

**Spherical Coordinates**

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right]\mathbf{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]\mathbf{a}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\mathbf{a}_\phi$$

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial\theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\mathbf{a}_\phi$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$