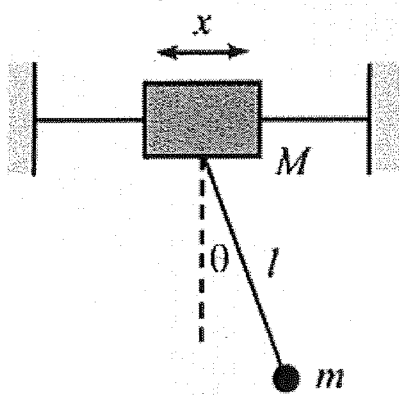
Ph. D. Comprehensive Examination
Spring 2013
Classical Mechanics 500

Consider a pendulum of length l and mass m that hangs from a mass M. The entire system can slide along a rail without any friction.



a) Write the Lagrangian as a function of x, θ (the angle of the pendulum), M and m.

b) Derive the equations of motion.

c) Considering only small oscillations find a conserved quantity associated with the motion.

d) Considering only small oscillations describe in words what happens to the system in the two normal modes.

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Classical Mechanics 600-1

Consider the Hamiltonian

$$H = \frac{1}{2}[p^2 + K^2 q^2]$$

where K is a constant.

a) Find the equations of motion for (q, p).

b) Verify that the following coordinate transformation is a canonical transformation:

$$q = \sqrt{\frac{2Q}{K}}\cos P \ , p = \sqrt{2QK}\ \sin P\ .$$

c) Find the transformed Hamiltonian using the transformations of (b).

d) Find the equations of motion for (Q, P).

(Hint: parts (c) and (d) can be done independently of part (b).)

Consider the central-force problem for an attractive potential of the form $V(r) = -\frac{k}{r^n}$, where k and n are two given real positive numbers.

a) Recalling the derivation of the equivalent one-dimensional problem, sketch the effective potential $W(r) = \frac{L^2}{2\mu r^2} - \frac{k}{r^n}$ as a function of r for two typical cases with $0 < n < 2$ and $n > 2$. What do L and $\frac{L^2}{2\mu r^2}$ represent physically?

b) Write the effective force equation corresponding to $\mu\ddot{r}$. For what values of r_0 and energy E_0 are circular orbits possible?

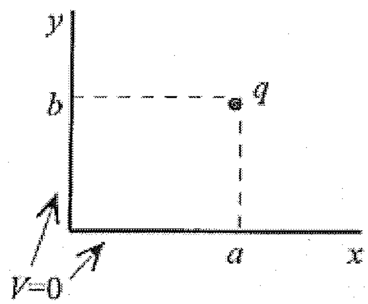
c) What is the period T_0 or the angular frequency $\omega_0 = \frac{2\pi}{T_0}$ of revolution for circular orbits, in terms of r_0?

d) Now consider a small perturbation from a circular orbit, inducing a small energy change from E_0. In which case do the perturbed orbits remain stable and nearly circular? What happens in the other cases?

e) Expand $r \approx r_0 + \eta$ in the effective force equation for stable nearly circular orbits, and obtain the frequency, Ω for small oscillations of r near r_0, in terms of r_0.

f) Determine the ratio of $\frac{\Omega}{\omega_0} = \frac{T_0}{T_r}$ in terms of n only. For which types of values of n do the nearly circular orbits remain closed (to this order of approximation)?

Two semi-infinite grounded conducting planes meet at right angles. In the region between them, there is a point charge q at position (a,b).



a) Set up the image configuration and calculate the potential in this region.

b) What is the force on q? How much work does it take to bring q in from infinity?

c) Suppose the planes met at some angle other than 90°, would you still be able to solve the problem by the method of images? If not, for what particular angles does the method work?

A plane polarized electromagnetic wave of frequency ω in free space is incident normally on the flat surface of a nonpermeable medium of conductivity σ and dielectric constant ϵ. Note that the nature of reflection and refraction in a conducting medium differ significantly from that in non-conducting media. Remember that the wave number (k) in the conducting medium is complex and can be expressed as $k = (\epsilon\mu)^{1/2}(\omega)(\alpha + i\beta)$, where

$\alpha = \frac{1}{\sqrt{2}}[(1 + (\frac{\sigma}{\omega\epsilon})^2)^{1/2} + 1]^{1/2}$,

and

$\beta = \frac{1}{\sqrt{2}}[(1 + (\frac{\sigma}{\omega\epsilon})^2)^{1/2} - 1]^{1/2}$.

a) Determine the amplitude and phase of the reflected wave relative to the incident wave for arbitrary σ and ϵ. Note: start with setting up your boundary conditions. It will help to recall that the phase angle ϕ is the ARG of the ratio of the reflected to incident fields (and ϕ will be the arctangent of some function).

b) Consider the limiting case of a very good conductor and show that the reflection coefficient (ratio of reflected to incident intensity) is:

$R \approx 1 - 2(\omega/\mathrm{c})\delta$,

where $\delta = \sqrt{\frac{2}{\mu\sigma\omega}}$, is the "skin depth".

A non-conducting spherical shell of radius R has a surface charge density $\sigma(\theta) = \sigma_0 (1 + \cos\theta)$.

a) Using an expansion in Legendre polynomials, find expressions $\Phi_{in}(r,\theta)$ and $\Phi_{out}(r,\theta)$ for the electrostatic potential inside and outside the sphere. Show explicitly which terms in the series are zero, and solve for the non-zero coefficients.

b) Calculate the electric dipole moment of the shell by direct integration. Using the dipole moment, calculate the electric dipole potential outside the shell, and compare it with the corresponding term in the expansion of $\Phi_{out}(r,\theta)$ in part (a).

Ph. D. Comprehensive Examination
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Electricity and Magnetism 600-3

a) Write down Maxwell's equations in vacuum.

b) Starting from Maxwell's equations in vacuum, show that $\vec{E}$ and $\vec{B}$ satisfy the wave equation.

Now assume:

$$\vec{E}(r,\theta,\phi,t) = E_0 \frac{\sin\theta}{r}\left[\cos(kr-\omega t) - \frac{\sin(kr-\omega t)}{kr}\right]\hat{\phi}, \text{ with } \frac{\omega}{k} = c.$$

[This is, incidentally, the simplest possible spherical wave. For notational convenience, let $(kr - \omega t) \equiv u$ in your calculations.]

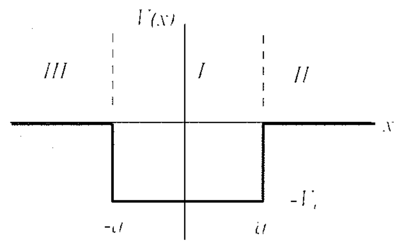
c) Verify that this $\vec{E}$ obeys all four Maxwell's equations in vacuum and find the associated magnetic field.

d) Calculate the Poynting vector $\vec{S}$. Average $\vec{S}$ over a full cycle to get the intensity vector $\vec{I}$. Does it fall off like r^{-2}, as it should?

e) Integrate $\vec{I} \cdot d\vec{a}$ over a spherical surface to determine the total power radiated.

A particle of mass m is bound by a one-dimensional finite square well potential

$$V(x) = -V_0 \qquad -a \le x \le a \quad \text{(Region I)}$$
$$ = 0 \qquad |x| > a \quad \text{(Regions II and III)}$$



a) Assume the particle has energy E, where $-V_0 < E < 0$.
(Note that, unlike the case for the infinite square well, the allowed energy values E are not given by a simple formula.)
Starting with the Schrödinger equation, write the spatial wave function $\psi(x)$ (with arbitrary constant coefficients) in each of the three regions *I, II* and *III*. Since $V(x)$ is a symmetric potential, solutions should be even or odd functions of x. If you eliminate any possible terms from your solution, explain why you do so.

b) **Sketch** the wave functions of the three lowest-energy states (assuming that their forms inside the well are qualitatively similar to those of the infinite well.)

c) Write the set of simultaneous equations which must be solved in order to find the energy eigenvalues E_n for this problem. (Hint: Consider the boundary conditions at $x = \pm a$.) Do not solve them.

For parts (d) and (e), assume that the two lowest energy eigenvalues of this system are E_1 and E_2, corresponding to normalized spatial wave functions $\psi_1(x)$ and $\psi_2(x)$.

At $t = 0$, the system is in the initial state $\Psi(x,0) = \frac{1}{2}\psi_1(x) - \frac{\sqrt{3}}{2}\psi_2(x)$.

d) Write a complete expression for $\Psi(x,t)$.

e) Show that the probability distribution for finding the particle at any x is a function of time, and find its frequency.

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Quantum Mechanics 600-1

A particle of mass m in a harmonic oscillator potential $V(x) = \frac{m\omega^2 x^2}{2}$ is subject to a small perturbing potential $v(x) = \lambda x$.

a) Derive the exact energy spectrum and eigenkets of the new Hamiltonian.

b) Using perturbation theory, evaluate the first- and second-order shifts in energy relative to the unperturbed Hamiltonian and compare your results to the exact answer in part (a).

(Hint: You may want to use: $\hat{A} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\frac{\hat{p}}{\sqrt{2m\hbar\omega}}$ to evaluate the matrix elements.)

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Quantum Mechanics 600-2

A particle has total angular momentum vector $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$, where $\boldsymbol{L}$ is the orbital angular momentum and $\boldsymbol{S}$ is the spin angular momentum. The magnitudes of $\boldsymbol{J}, \boldsymbol{L}$ and $\boldsymbol{S}$ are characterized by the quantum numbers l, s and j, and their projections along the z-axis are defined by the quantum numbers m_j, m_l and m_s.

a) Write $\langle \boldsymbol{L}\cdot\boldsymbol{S} \rangle$, the expectation value of the dot product $\boldsymbol{L}\cdot\boldsymbol{S}$, in terms of the quantum numbers l, s and j. (Hint: take the dot product $\boldsymbol{J}\cdot\boldsymbol{J}$.)

b) Find the possible values of j and of $\langle \boldsymbol{L}\cdot\boldsymbol{S} \rangle$ when $l = 2$ and $s = 1/2$.

An external magnetic field $\boldsymbol{B}$ is applied in the z-direction. The energy of an electron shifts by an amount $\Delta E = -\boldsymbol{\mu}\cdot\boldsymbol{B}$, where $\boldsymbol{\mu}$ is the magnetic dipole moment of the electron. The magnetic dipole moment of an electron with orbital angular momentum $\boldsymbol{L}$ and spin $\boldsymbol{S}$ is given by

$$\boldsymbol{\mu} = -\frac{\mu_B}{\hbar}(\boldsymbol{L} + 2\boldsymbol{S}), \text{ where } \mu_B \text{ is the Bohr magneton } (\mu_B = e/2m).$$

c) Noting that $\boldsymbol{L} + 2\boldsymbol{S} = \boldsymbol{J} + \boldsymbol{S}$, and using the fact that $\langle \boldsymbol{S} \rangle = \dfrac{\langle \boldsymbol{S}\cdot\boldsymbol{J} \rangle}{\langle \boldsymbol{J}\cdot\boldsymbol{J} \rangle}\langle \boldsymbol{J} \rangle$, calculate the **energy shift** ΔE in terms of B, μ_B and the quantum numbers l, s, j and m_j.

In all of the above equations, the symbols $\boldsymbol{L}, \boldsymbol{S}, \boldsymbol{J}, l, s, j$ can equally well refer to the **total** orbital and spin angular momenta of **all** the "active" electrons (usually the valence electrons) in an atom. Call these total quantum numbers l', s' and j'.

Note: Part (d) does not depend on parts (a)-(c).

d) Consider an atom with 2 electrons in a "d-shell" (i.e., $l = 2$ for each electron), and no other active electrons. Make a table with columns labeled "l' s' j'", and for each possible combination of l' and s' (the total orbital and spin quantum numbers for the 2-electron system) list all possible values of j'.

e) Using the results of (c) and (d), calculate the largest possible energy shift ΔE for this atom in an external magnetic field B when j' takes its maximum possible value.

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Quantum Mechanics 600-3

Consider the 2-dimensional Schrödinger equation (SE) with a potential

$$V(x, y) = \frac{1}{2} k \frac{x^4 + y^4}{x^2 + y^2}.$$

In polar coordinates, the potential becomes:

$$V(r, \theta) = \frac{1}{4} kr^2 [1 + cos^2(2\theta)],$$

since $x = r\ cos\theta$ and $y = r\ sin\theta$, $0 \leq \theta < 2\pi$. We wish to obtain an upper estimate of the ground-state energy of a particle of mass m in that potential using a trial (non-normalized) wavefunction

$$\varphi(x, y) = exp\left[-\frac{1}{2}\frac{m\varpi}{\hbar}(x^2 + y^2)\right],$$

where ϖ is a variational parameter.

a) Explain the rationale and the basic assumption and approximation that is used in choosing such a trial wavefunction $\varphi(x, y)$.

b) Determine the normalization $\langle \varphi | \varphi \rangle$ and use it subsequently.

c) Show that the average kinetic energy is $\langle T \rangle_\varphi = \frac{1}{2}\hbar\varpi$. You may use specific theorems and arguments that apply to this case, or compute $\langle T \rangle_\varphi$ explicitly, preferably using polar coordinates, in which

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\varphi(r) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\varphi(r)\right).$$

You may use the integral formula

$$I_n = \int_0^\infty e^{-\alpha y^2} y^n dy = \frac{1}{2\alpha^{(n+1)/2}}\Gamma\left(\frac{n+1}{2}\right).$$

d) Compute the average potential energy $\langle V \rangle_\varphi$ using polar coordinates.

e) Determine $\overline{\omega}$ by minimizing $\langle T \rangle_{\varphi} + \langle V \rangle_{\varphi}$.

f) Is angular momentum $L_z = -i\hbar \frac{\partial}{\partial \theta}$ conserved for this potential? Explain.

An Otto cycle consists of the following four stages (consider as moving through points a, b, c, and d)

$a \rightarrow b$, Compression stroke (adiabatic compression)
$b \rightarrow c$, Ignite fuel (heating at constant volume)
$c \rightarrow d$, Power stroke (adiabatic expansion)
$d \rightarrow a$, Reject heat to environment (cooling at constant volume).

a) Explain what occurs within the gas during the adiabatic transitions.

b) Draw the PV diagram for an idealized Otto cycle. Note positions of minimum and maximum volume, V_1 and V_2, respectively.

c) Draw the TS diagram for an idealized Otto cycle. Note positions of minimum and maximum entropy, S_1 and S_2, respectively.

d) Show that (i.e., derive it) the efficiency, ϵ, of the Otto cycle for an ideal gas (with temperature-independent heat capacities) as a function of the ratio V_1/V_2 and the heat capacity per particle C_V is:

$$\epsilon = 1 - \left(\frac{V_1}{V_2}\right)^{-Nk/C_V}$$

where N is number of molecules and k is Boltzmann's constant.

Consider a single spin σ_1 taking values +1 or -1 in a constant external magnetic field. The energy is given by:

$$E = -h\sigma_1$$

where h is a coupling constant.

a) Evaluate the partition function, $Z_1(T, h)$, for this system.

b) Find $\langle E \rangle$.

c) Find $\langle \sigma_1 \rangle$.

Now consider two spins σ_1 and σ_2, each taking values +1 or -1, coupled together by the Ising interaction and placed in a constant external magnetic field along the z-axis. The energy becomes:

$$\beta E = -h(\sigma_1 + \sigma_2) - K\sigma_1\sigma_2 .$$

Here K and h are the coupling constants: K couples the spins to each other, h couples them to the external magnetic field, and $K > 0$.

d) How many states does the system have? Draw a figure giving all the possible spin configurations.

e) Evaluate the partition function Z_2.

f) Using

$$\frac{\partial \ln Z_2}{\partial K} = \langle \sigma_1\sigma_2 \rangle \quad \& \quad \frac{\partial \ln Z_2}{\partial h} = \langle \sigma_1 + \sigma_2 \rangle$$

show that:

$$\langle \sigma_1\sigma_2 \rangle = \frac{\cosh(2h) - e^{-2k}}{\cosh(2h) + e^{-2k}} \quad \& \quad \langle \sigma_1 + \sigma_2 \rangle = \frac{2\sinh(2h)}{\cosh(2h) + e^{-2k}} .$$

Consider a classical ideal gas consisting of N identical particles in three dimensions confined to a volume V in thermal equilibrium at temperature T. Each particle is labeled by its position and momentum $(\vec{r}_i, \vec{p}_i)$ where $i = 1, 2, 3, \ldots, N$. Since the particles do not interact with each other, the energy of the system is given by:

$$E = \sum_{i=1}^{N} \frac{|\vec{p}_i|^2}{2m}$$

where $|\vec{p}_i|^2 = p_{ix}^2 + p_{iy}^2 + p_{iz}^2$.

(a) Evaluate the canonical partition function for this system.

(b) Find its free energy.

(c) Verify the ideal gas equation of state.

(d) Find the average energy $\langle E \rangle$.

(e) Find an expression for the joint probability that particle 1 has a speed between v_1 and $v_1 + dv_1$, particle 2 between v_2 and $v_2 + dv_2$, and so on: $P(v_1, v_2, \ldots, v_N)\, dv_1\, dv_2 \ldots\, dv_N$. Don't forget to include the proper normalization. (Hint: Starting from the Boltzmann distribution, first, carry out the trivial integration over position coordinates; then convert to spherical polar coordinates in momentum space and integrate over the angles; and then switch from the magnitude of momentum to speeds.)

(f) Using the fact that particle speeds in an ideal gas are stochastically independent, find the (normalized) probability $P(v)\, dv$ that any molecule has a speed between v and $v+dv$. This is the Maxwell speed distribution.