Ph.D. Comprehensive Examination

Physics Department

Spring 2010

Thursday, March 18, and Friday, March 19, 2010

Room 133 Hannan Hall

GENERAL INSTRUCTIONS:

This examination is divided into four sections as follows:

<u>Thursday, March 18, 2010</u>

9:00 am – 12:00 Noon	Classical Mechanics – **2 questions**
1:00 pm – 5:00 pm	E & M – **3 questions**

<u>Friday, March 19, 2010</u>

9:00 am – 12:00 Noon	Stat Mech. – **2 questions**
1:00 pm – 5:00 pm	Quantum Mech. – **3 questions**

In **each** of the four subject areas, you may answer the **500-level** question in place of a 600-level question.

<u>DO EACH PROBLEM IN A SEPARATE BLUE BOOK</u>

PUT YOUR NAME ON EACH BOOK

LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS
FOR EXAMPLE: Mechanics 600-1

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RULES FOR THE PHYSICS COMPREHENSIVE EXAMINATION

Spring 2010 (March 18 & 19, 2010)

In order to assure an equitable and fair Comprehensive Examination for all students the following rules will apply to the examination administered by the Department of Physics.

1. The examination will be **closed book.** Students will not be permitted to bring to the examination room materials such as notes, reference books, calculators, or other aids of any form. The Physics Department will supply copies of the *CRC Mathematical Handbook,* Schaum's *Mathematical Handbook,* the *Table of Functions* by Jahnke and Emde, and the *NBS Handbook of Mathematical Functions,* for use during the examination.

   Students who want to use their own mathematical tables and/or handbooks must get permission from the Chairman prior to the beginning of the comprehensive examination. These tables and/or handbooks will remain under the control of the department until after the examination is completed, but will be available to their owners during the examination period.

   The Physics Department will supply calculators for use during the examination.

2. If needed, students will be allowed a short break, not to exceed ten minutes, during the examination period. While absent from the examination room students may not consult material or other sources of information concerning any matter pertaining to the examination. Students who leave the examination room will turn in all of their examination material to the proctor before leaving the room. Examination papers will be returned to the students when they reenter the examination room. Students who fail to return within ten minutes will be considered to have completed that portion of the examination.

   Only one student will be permitted to leave the examination room at a time.

A particle of mass m moves in a singular central potential $V(r) = -\frac{K}{r^n}$ with $n > 2$ and $K > 0$.

(a) What is the effective potential $V_{eff}(r)$ of the equivalent one-dimensional problem? Sketch $V_{eff}(r)$ for a given $L > 0$. What is L?

(b) For a given $L > 0$, determine the radius r_0 of a circular orbit. Is this orbit stable or unstable? What is the particle energy E_0 for this orbit?

(c) Discuss the nature of orbits with energies $E > E_0$. Are they bound or unbound? Are they captured by the singularity?

(d) Discuss the nature of orbits with $0 < E < E_0$. What types of turning-points do they have? Are they captured by the singularity?

(e) For bound orbits, determine whether the particle takes a finite or an infinite time to spiral into the singularity. Correspondingly, determine whether the particle passes through a finite or an infinite number of revolutions while spiraling into the singularity.

Consider a collection of particles whose position vectors $\vec{r}_i$ and momenta $\vec{p}_i$ are both bounded (i.e., remain finite for all time). Define a quantity:

$$S \equiv \sum_i \vec{p}_i \cdot \vec{r}_i$$

(a) Starting by taking the time derivate of S, determine the average value of dS/dt over a time interval τ. Note that if the system's motion is periodic, and τ is some integer multiple of the period, $\left\langle \frac{dS}{dt} \right\rangle$ will vanish. However, even if the system does not display any periodicity, we can make $\left\langle \frac{dS}{dt} \right\rangle$ as small as desired by making τ sufficiently long. Show, in this limit that:

$$\langle K \rangle = -\frac{1}{2}\left\langle \sum_i \vec{F}_i \cdot \vec{r}_i \right\rangle,$$

where $\langle K \rangle$ is the average total kinetic energy of the system and $\vec{F}_i$ is the force on the i^{th} particle. This is the ***Virial Theorem.***

(b) Consider an ideal gas containing N atoms in a container of volume V, pressure P, and absolute temperature T. The only force involved is the force of constraint by the walls of the container. Use the virial theorem to derive the equation of state for a perfect gas:

$$NkT = PV$$

where k is Boltzmann's constant.

Note, ac cording to the equipartition theorem, the average kinetic energy of each atom in the ideal gas is 3/2 kT

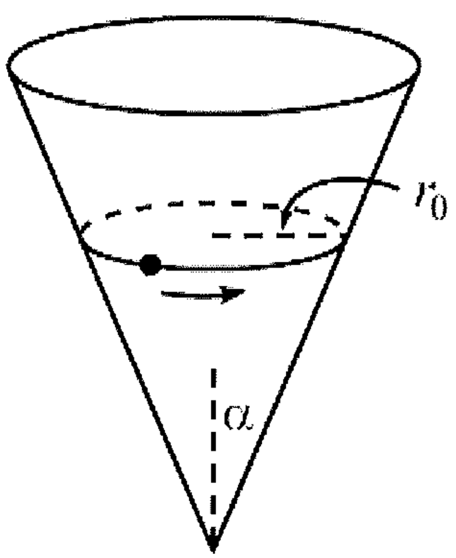
PhD/MS Comprehensive Examination
Spring 2010
Classical Mechanics 500-1

A particle slides on the inside surface of a frictionless cone, as shown in the diagram. The cone is fixed with its tip on the ground and its axis vertical. The half-angle α is shown in the diagram. Let $r(t)$ be the distance from the particle to the axis, and $\theta(t)$ be the angle around the cone.

(a) Find the equations of motion.

(b) If the particle moves in a circle of radius r_0 what is the frequency ω of this motion?

(c) If the particle is perturbed from this circular motion, what is the frequency Ω of the oscillations about the radius r_0.



Consider an iron sphere of radius R that carries a charge Q and a uniform magnetization $\vec{M} = M\hat{z}$. The sphere is initially at rest.

(a) Show that the magnetic field outside the uniformly magnetized sphere is

$$\vec{B}(\vec{r}) = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right),$$

where m is the magnetic dipole moment.

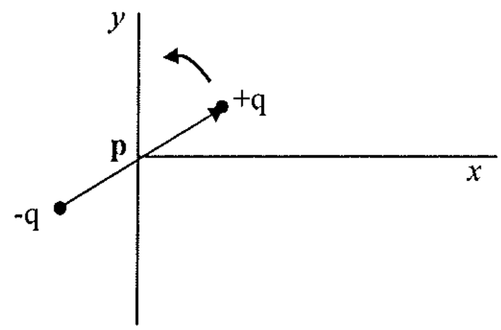
(b) Compute the angular momentum stored in the electromagnetic field.

(c) The sphere is gradually and uniformly demagnetized by heating it up past the Curie point. Find the induced electric field.

(d) Find the torque that the induced electric field exerts on the sphere.

(e) Calculate the total angular momentum imparted to the sphere after its complete demagnetization.

An electric dipole **p** (vector) lies in the x, y-plane and is rotating with an angular velocity ω about the z-axis, as shown in figure below.



(a) Find the electric and magnetic fields of this rotating dipole.

(b) Find the Poynting vector and the intensity of radiation.

(c) Sketch the intensity profile as a function of the polar angle θ.

(d) Calculate the power radiated.

Consider a point-particle of charge q moving with constant velocity $\vec{v}$ in the Laboratory Frame K, where events have Lorentz coordinates $x^{\mu} = (ct, x, y, z)$. Consider also the Particle Frame K', where the particle is at rest at the origin, and events have Lorentz coordinates $x^{\mu'} = (ct', x', y', z')$.

(a) Write the Lorentz Transformations between coordinates in the two Frames K and K', assuming that $\vec{v} = v\hat{e}_x$, say.

(b) Determine a 4-vector potential $A^{\alpha'}(x^{\mu'}) = (\phi', \vec{A}')$ in the Particle Frame K′.

(c) Determine the corresponding $A^{\alpha}(x^{\mu})$ in the Laboratory Frame K.

(d) Determine the electric and magnetic fields $\vec{E}'$ and $\vec{B}'$ in the Particle Frame K′.

(e) Determine the electric and magnetic fields $\vec{E}$ and $\vec{B}$ in the Laboratory Frame K.

(f) Verify that the general Field Transformations

$$\vec{E}_{//} = \vec{E}'_{//}$$
$$\vec{E}_{\perp} = \gamma(\vec{E}'_{\perp} - \frac{1}{c}\vec{v} \times \vec{B}')$$
$$\vec{B}_{//} = \vec{B}'_{//}$$
$$\vec{B}_{\perp} = \gamma(\vec{B}'_{\perp} + \frac{1}{c}\vec{v} \times \vec{E}')$$

are satisfied in this problem.

(g) Consider the non-relativistic limit $v << c$. Show that $\vec{B} \cong \frac{q}{c}\vec{v} \times \frac{\vec{r}}{|\vec{r} - \vec{v}t|^3}$ in the Laboratory Frame K and interpret this result in light of the Biot-Savart Law.

A metal cross bar of mass m slides without friction on two long parallel conducting rails that are separated by a distance b. A resistor, R, is connected between the rails at one end; compared to R, the resistances of the bar and rails are negligible. There is a uniform field $\vec{B}$ perpendicular to the plane containing the rails and the bar (i.e., into the page in the figure below). At time $t = 0$, the crossbar is given a velocity $\vec{v}_0$ toward the right. Subsequently:

(a) What is the direction and magnitude of the electric field $\vec{E}$ that the bar sees in its rest frame?

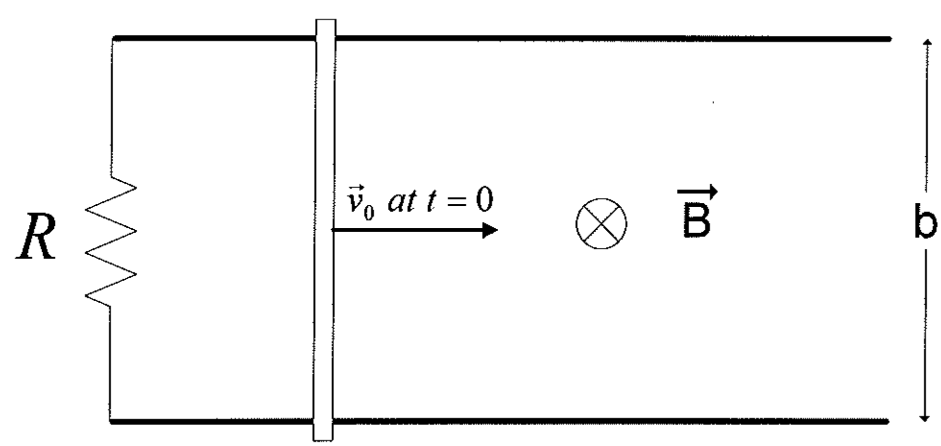
(b) Find the current in the bar.

(c) Find the force on the bar and, therefore, its equation of motion.

(d) Find the time at which the bar comes to rest.

(e) Find the total distance the bar travels.

(f) Confirm that energy is conserved in this process. (Where does the initial kinetic energy of the bar go?)



Consider an ideal gas of quantum particles obeying Bose-Einstein statistics. The system has volume $V_n = L^n$ in n dimensions and its particles have an ultra-relativistic linear ($s = 1$) or non-relativistic quadratic ($s = 2$) energy-momentum relation $\varepsilon = C|\vec{p}|^s$.

(a) Determine the n- and s-dependence of the particle density of states $g(\varepsilon)d\varepsilon$.

(b) Provide the expressions for the average number of particles N and the internal energy U in terms of $\beta = 1/kT$ and fugacity $z = e^{\mu/kT}$.

(c) Provide the expression for the pressure P in terms of β and z.

(d) Find the relation between U and PV_n in terms of n and s. This relation should be proved from the expressions provided in (a – c).

(e) Provide the criterion for yielding Bose-Einstein condensation (BEC). In which of the four cases with $n = 2, 3$ and $s = 1, 2$ can BEC occur?

Consider a dilute gas that can be treated as ideal. The molecules of this gas can exist in two forms, A and B, that can interconvert, $A \leftrightarrow B$, such that the two forms exist as an equilibrium mixture. According to the Boltzmann distribution law, the equilibrium ratio of A and B populations is given by

$$\frac{\langle N_A \rangle}{\langle N_B \rangle} = \frac{g_A}{g_B} e^{-\beta \Delta \varepsilon}$$

where $\Delta\varepsilon$ is the energy difference between state A and state B, and g_A and g_B are their respective degeneracies.

(a) Derive this result from the condition for equality of the chemical potentials at equilibrium.

The canonical partition function is $Q = q^N / N!$, where N is the total number of particles and q is the Boltzmann weighted sum over *all* single molecule states, both those associated with A and those associated with B.

(b) Show that $Q = \sum_P \exp[-\beta A(N_A, N_B)]$, with $-\beta A(N_A, N_B) = \ln[q_A^{N_A} q_B^{N_B} / (N_A! N_B!)]$, and where the summation is over all partitions of the N molecules into N_A of type A and N_B of type B.

(c) Show that the condition for chemical equilibrium in terms of the chemical potentials is equivalent to minimizing the Helmholtz free energy $\partial A / \partial \langle N_A \rangle = \partial A / \partial \langle N_B \rangle = 0$ subject to the constraint that $\langle N_A \rangle + \langle N_B \rangle = N$ is fixed.

(d) Show that $\langle N_A \rangle = q_A (\partial \ln Q / \partial q_A)_{q_B, N} = N q_A / (q_A + q_B)$ and hence, that $\langle N_A \rangle / \langle N_B \rangle = q_A / q_B$.

(e) Show that the mean square fluctuation in N_A is given by

$$\langle [N_A - \langle N_A \rangle]^2 \rangle = q_A (\partial \langle N_A \rangle / \partial q_A)_{q_B, N} = \langle N_A \rangle \langle N_B \rangle / N .$$

For a certain substance near its solid-liquid phase transition, the Helmholtz free energy per unit volume is given by

$$\frac{A_l}{V} = \frac{\alpha\rho^2}{2T} \text{ in the liquid phase, and}$$

$$\frac{A_s}{V} = \frac{\beta\rho^3}{3T} \text{ in the solid phase,}$$

where $\rho = n/V$ is the molar density and the subscripts l and s refer to the liquid and solid phases, respectively. At a given temperature, the pressure can be adjusted to a particular pressure P_f, at which point the liquid freezes. Just before solidification, the liquid density is ρ_l and just after solidification the solid density is ρ_s.

(a) By considering the condition on the chemical potentials for equilibrium of the two phases (coexistence), determine ρ_l and ρ_s as functions of temperature.

(b) By considering the condition on the pressures for equilibrium of the two phases (coexistence), determine P_f as a function of temperature.

(c) Determine the change in entropy per mole in the process of solidification.

(d) Determine the change in volume per mole in the process of solidification from your results in part (a).

(e) Show that these results satisfy the Clausius-Clapeyron equation:

$$\frac{\Delta s}{\Delta v} = \frac{dp}{dT} \text{ along a coexistence curve.}$$

A spinless particle of mass m and charge q, in a three-dimensional harmonic oscillator potential and a uniform magnetic field $\vec{B} = B\hat{z}$, is governed by the Hamiltonian:

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + \frac{m}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right)$$

In the gauge $\vec{A} = \vec{B} \times \vec{r}/2$ the first term takes the form

$$\frac{p^2}{2m} - \frac{q}{2mc}\vec{L}\cdot\vec{B} + \frac{q^2 B^2}{8mc^2}\left(x^2 + y^2\right)$$

Suppose the oscillator is anisotropic, with frequencies

$$\omega_x = \omega + \Delta, \quad \omega_y = \omega, \quad \omega_z = \omega, \qquad with\ 0 < \Delta < \omega .$$

(a) If B=0, what are the three lowest energy eigenvalues and their degeneracies?

(b) Specify the condition on B, in terms of the parameters in the Hamiltonian, under which the B^2 term is negligible compared with the term linear in B, for the low lying states with $\vec{L} \neq 0$?

For the rest of the problem, neglect the B^2 term.

(c) Assuming the remaining magnetic interaction energy from the $\vec{L}\cdot\vec{B}$ term is small compared to $\hbar\Delta$, find the lowest non-vanishing perturbation (if any) of the three lowest energy eigenvalues.

(d) Now consider the isotropic (Δ=0) oscillator as the unperturbed system. Treating both the anisotropy and the magnetic term as perturbations, without assuming the magnetic energy is small compared to $\hbar\Delta$, find the lowest non-vanishing perturbation (if any) of the first excited level of the isotropic oscillator.

(e) Sketch a graph of the perturbed levels, arising from the first level of the isotropic oscillator, as a function of B with Δ fixed, showing clearly the asymptotic behavior for small and large B.

Useful Formulas:

$L_z = i\varsigma\hbar(a_y^\dagger a_x - a_x^\dagger a_y)$, where the operators on the right hand side are raising and lowering operators for the x and y modes of the oscillator, and the numerical factor, ζ, is given by

$$\varsigma = \frac{1}{2}\left(\sqrt{\frac{\omega_x}{\omega_y}} + \sqrt{\frac{\omega_y}{\omega_x}}\right)$$

Although no experimental evidence exists for it, many present-day theories of particle physics predict in nature a spin-3/2 particle called the "gravitino". A set of spin matrices must satisfy the following fundamental relations:

$$[S_x, S_y] = i\hbar S_z, \quad [S_y, S_z] = i\hbar S_x, \quad [S_z, S_x] = i\hbar S_y, \tag{1}$$

along with the eigenvalue equations,

$$\begin{aligned} S^2 \mid s,m> &= \hbar^2 s(s+1) \mid s,m> \\ S_z \mid s,m> &= \hbar m \mid s,m> \end{aligned} \qquad (2)\ \&\ (3)$$

acting on the orthonormal states $| s,m >$. Furthermore, we may define raising and lowering operators $S_\pm \equiv S_x \pm iS_y$ so that

$$S_\pm \mid s,m >= \hbar\sqrt{(s \mp m)(s \pm m+1))} \mid s,m \pm 1 >, \tag{4}$$

where $S_\pm \equiv S_x \pm iS_y$. For a spin 3/2 particle, $s = 3/2$.

(a) Give a set of 4x1 matrices that represent the respective states $| \frac{3}{2}, \frac{3}{2} >$, $| \frac{3}{2}, \frac{1}{2} >$, $| \frac{3}{2}, -\frac{1}{2} >$, and $| \frac{3}{2}, \frac{3}{2} >$. Using (3) write the corresponding 4x4 matrices S_z.

(b) By taking the expectation value of (4) with $< \frac{3}{2}, m'|$, find 4x4 matrices S_x and S_y. Verify that these satisfy (1).

(c) If we prepare a beam of gravitinos in the linear combination of states given by $\frac{1}{\sqrt{2}}[| \frac{3}{2}, \frac{3}{2} > + | \frac{3}{2}, \frac{1}{2} >]$, what is the expectation value $< \vec{S} >$?

(d) If the Hamiltonian is of the form $(E_0/\hbar^2) | \vec{S} |^2 + \gamma \vec{B}_0 \cdot \vec{S}$, what are the energy levels for the gravitino beam?

Consider two inertial reference frames K and K' moving with a constant velocity $\vec{V}$ relative to each other.

(a) Write the Galilean Transformation between the coordinates $x^{\mu} = (t, x, y, z)$ and $x^{\mu'} = (t', x', y', z')$ of events in the frames K and K', assuming a non-relativistic speed $V << c$ in that transformation.

(b) Assume that the Schroedinger equation (SE) for a single quantum mechanical particle of mass m is invariant in form under any Galilean Transformation. Furthermore, assume that $\Psi(\vec{x}, t)$ and $\Psi'(\vec{x}', t')$ are free-particle solutions of the SE in K and K', respectively. Thus find the Transformation Law for wave-functions in a Galilean Transformation.

(c) Now assume that the particle may be subject to a potential energy

$$U(\vec{x}, t) = U'(\vec{x}', t')$$

invariant in value in K and K'. What would that mean for a harmonic oscillator potential, for example?

(d) Verify that the SE is indeed generally invariant in form with the Transformation Law for the wave-function that you found. Assume $\vec{V} = V\hat{e}_x$, for simplicity, and use $\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}$, and $\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + V\frac{\partial}{\partial x}$ for the Galilean Transformation.

A particle of mass m moves in one dimension in the potential $V(x) = m\omega^2 x^2/2$. At time $t = 0$ the wave function of the particle is

$$\Psi(x,0) = A\sum_n \frac{\psi_n(x)}{2^{n/2}}$$

where the $\psi_n(x)$ are the eigenstates of the energy with eigenvalues $E_n = (n + \frac{1}{2})\hbar\omega$.

(a) Find the normalization constant A.

(b) Write out $\Psi(x,t)$ for $t > 0$.

(c) Show that the probability density $|\Psi(x,t)|^2$ is a periodic function of time and determine the longest possible period.

(d) Calculate the expectation value of the energy at $t = 0$ and show that it is equal to $3\hbar\omega/2$.

(e) What is the probability that a measurement of the energy at $t = 0$ would yield the value $3\hbar\omega$?

(f) What is the probability that a measurement of the energy at $t = 0$ would yield the value $5\hbar\omega/2$?