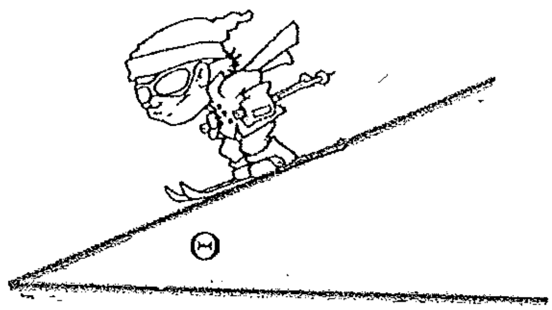
MS/PhD Comprehensive Exam
Spring 2009
Mechanics 500-1

A skier of mass m goes downhill at an angle of Θ to the horizontal. The drag force is proportional to the velocity ($F = -bv$).



(a) What is the skier's terminal velocity?

(b) If the skier starts from rest at t=0, find the velocity as a function of time.

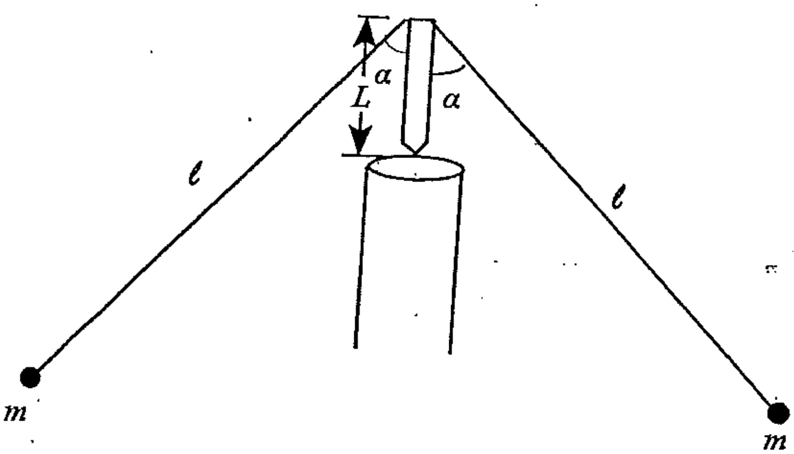
(c) After the skier approximately reaches his terminal velocity, how much energy does he lose per unit time due to the drag force?

(d) Where does this energy go?

(e) If the skier is losing energy, why doesn't he slow down further? i.e. why doesn't his velocity drop below terminal velocity?

PhD/ MS Comprehensive Examination
Spring 2009
Mechanics 500-2

A "teeter toy," shown in the diagram; consists of two identical masses m on the ends of rigid rods; the other ends of the rods are connected to a pin. The rods and the pin lie in a common plane. The rods are each of length l and each makes an angle α with the pin. The pin rests on a fixed pedestal, as shown.



Consider only rocking motions of the teeter toy in the plane of the diagram such that the plane of the toy remains vertical. Measure the displacement by the angle θ that the pin makes with the vertical. Take the zero of gravitational potential energy to be the point at which the pin meets the pedestal. Assume that the masses of the rods and the pin are negligible compared to m.

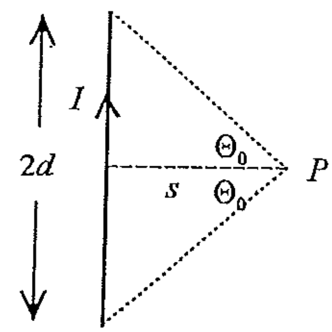
a) Show that the potential energy of the toy when it is displaced by an angle θ from the vertical is given by:

$$V = 2mg\,(L - l\cos\alpha)\cos\theta\,.$$

b) Find the equilibrium value of θ from this potential energy.

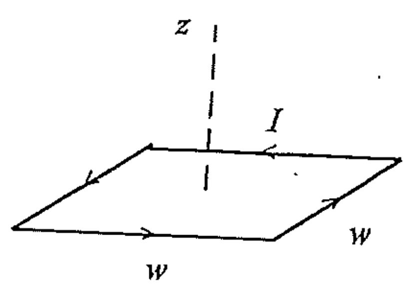
c) Find the condition such that this equilibrium position is stable.

a) Using Biot-Savart Law $d\vec{B} = \frac{\mu_0}{4\pi}\frac{Idl \times \hat{r}}{r^2}$, (where $\hat{r} = \frac{r}{r}$),
show that the magnetic field contribution, due to a finite straight segment of wire of length $2d$ carrying a current I, at a point P, which is located a distance s from the midpoint of the wire segment, is $B = \frac{\mu_0}{2\pi s} I \sin\theta_0$, where θ_0 is the angle subtended by each half of the wire segment at point P.



b) What is the direction of the magnetic field at point P?

c) Using the results of parts (a) and (b), find the magnitude and direction of the magnetic field $B(z)$ at an arbitrary point along the z-axis due to a square loop of wire, length w on a side, lying in the x-y plane centered on the origin and carrying a counter-clockwise current I.



PhD/ MS Comprehensive Examination
Spring 2009
Electricity and Magnetism 500-2

Consider a plane electromagnetic wave, propagating in vacuum, described by an electric field of the form:

$$E(x,t) = \hat{j}E_{\max}e^{i(kx-\omega t)}\text{, where } \hat{j} \text{ is a unit vector in the } y\text{-direction.}$$

a) What are the amplitude and direction of the magnetic field?

b) Based on the form of the plane wave, write expressions for the wavelength, frequency, and phase velocity.

c) Starting with the total energy density $u = \frac{1}{2}\varepsilon_o E^2 + \frac{1}{2\mu_0}B^2$, where ε_0 and μ_0 are, respectively, the permittivity and permeability of free space, show that the energy flow, dU, through an area A in a time dt, is $dU = (\varepsilon_0 E^2)(Ac\,dt)$.

d) Given the expression in (c), what is the energy flow per unit area, per unit time? In which direction is this energy transported? [Note, that this will be the magnitude of the Poynting vector in vacuum].

e) What is the average intensity, I, of this wave?

f) Now, consider a spherical wave emitted by a point source, propagating in vacuum with a transverse electric field of the form $E(r,t) = \frac{A}{r}e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, in the far region, where A is a corresponding constant. Determine the average rate at which energy is transported by the wave (i.e., the power).

a) One mole of an ideal gas is taken from (T_1, V_1) to (T_2, V_2), where T is the temperature and V is the volume. Starting from the first and second laws of thermodynamics show that the change in entropy is:

$$\Delta S = C_V \ln\left[\frac{T_2}{T_1}\right] + R\ln\left[\frac{V_2}{V_1}\right]$$

b) An ideal gas is expanded adiabatically from (P_1, V_1) to (P_2, V_2), where P is the pressure. It is then compressed at constant pressure to (P_2, V_1). Finally, the pressure is increased to P_1 at constant volume V_1.

i) Draw the cycle on a P-V plot.
ii) Show that the efficiency, η, of the cycle is:

$$\eta = 1 - \gamma\frac{(V_2/V_1)-1}{(P_1/P_2)-1}$$

where, $\gamma = C_P/C_V$

A building at a temperature T is heated by means of a heat pump, which uses a river at a temperature T_0 as a source of heat. The heat pump, which may be assumed to have an ideal performance, consumes a constant power W, and the building loses heat to its surroundings at a rate $\alpha(T - T_0)$, where α is a constant. Show that the equilibrium temperature of the building, T_e, is given by

$$T_e = T_0 + \frac{W}{2\alpha}\left\{1+\left(1+\frac{4\alpha T_0}{W}\right)^{\frac{1}{2}}\right\} .$$

A particle of mass m moves in one dimension in the potential

$$V(x) = 0 \quad \text{for} \quad 0 < x < a$$
$$V(x) = \infty, \quad \text{otherwise.}$$

a) Write out and solve the Schrodinger equation for this system and find the allowed energies and normalized eigenfunctions.

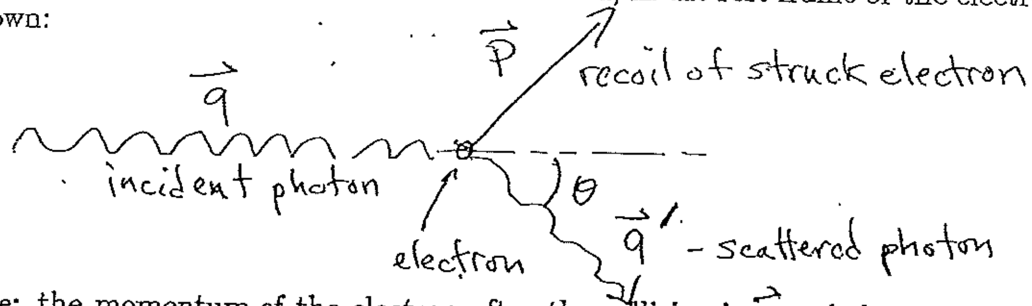
Assume now that at time $t = 0$ the particle is in the state

$$\Psi(x,0) = \sqrt{\frac{8}{5a}}\left[1 + \cos(\pi x / a)\right]\sin(\pi x / a).$$

b) If the energy of the particle is measured at $t = 0$, what are the possible results and what is the probability of each?

c) Find $\Psi(x,t)$.

1. Consider an atomic gas, containing some free electrons. Now, consider the collision between an energetic photon and a free electron, in the rest frame of the electron, as shown:



Note: the momentum of the electron after the collision is $\vec{p}$ and the momentum of the photon before the collision is $\vec{q}$ and, after the collision, is $\vec{q}'$.

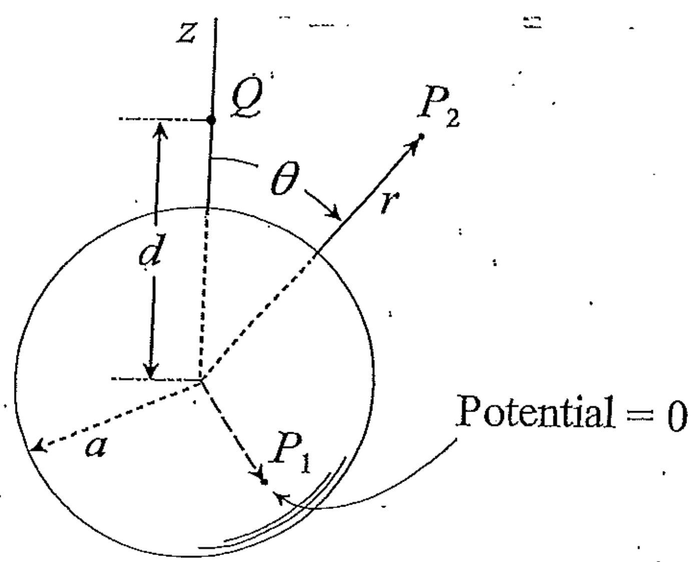
a) using Conservation of Energy and Conservation of Momentum, show that change in the wavelength of the scattered photon is given as: $\lambda' - \lambda = (h/m_e c)(1 - \cos(\theta))$. [Note: frequency = c/λ].

b) In a Compton Scattering experiment, an incoming X-ray, of wavelength $\lambda = 5.53 \times 10^{-2}$ nm enters a gas and is scattered and deflected at an angle of 35^0. The fractional shift in wavelength of the scattered X-ray is found to be about 1% of its initial wavelength (an easily measurable shift). However, in this experiment some of the deflected X-rays seem not to have had their wavelengths shifted. Explain what might have occurred.

c) For what Kinetic Energy will a particle's de Broglie wavelength equal its Compton wavelength?

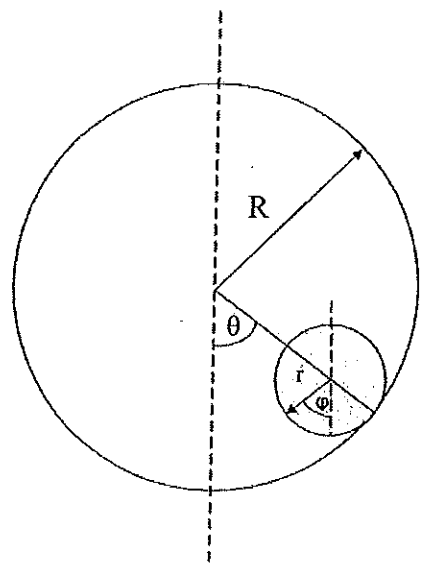
Consider a point charge, Q, at a distance d from the center of a grounded conducting sphere of radius a (see the figure below). Using the method of images:

(a) Find the magnitude and position of the image charge.
(b) Show explicitly that the resulting charge arrangement will make the potential zero at a general point P_1 on the surface of the sphere (as shown).
(c) Determine the potential at an arbitrary point P_2 outside the sphere at a position (r, θ).
(d) Find the electric field at the point P_2.
(e) Determine the induced charge density, σ, on the surface of the sphere.
(f) Now, consider the case when the sphere is at a potential V other than zero. What are the position(s) and magnitude(s) of the image charges?



A solid homogeneous cylinder of radius r and mass m **rolls without slipping** on the inside of a larger cylinder of radius R, as shown in the diagram. The larger cylinder is fixed in place and is *not* free to move.

a) If the small cylinder starts at rest at an angle $\theta = \theta_0$ from the vertical, find the total downward force it exerts on the outer cylinder as it passes through the lowest point.
b) Using the coordinates shown in the diagram, find the Lagrangian.
c) Find the equation of motion.
d) Find the frequency of small oscillations about the position of stable equilibrium.



Consider a point-particle of mass m in an attractive central force field $\vec{F} = -\frac{k}{r^3}\vec{r}$, where k is a positive constant. Define the Lenz vector as

$$\vec{A} = \frac{\vec{r}}{r} - \frac{1}{mk}\vec{p} \times \vec{L},$$

where $\vec{r}$, $\vec{p}$, and $\vec{L}$ denote the position, momentum, and angular momentum of the particle.

a) Show that $\vec{A}$ is a constant of motion, i.e., $\frac{d\vec{A}}{dt} = 0$.

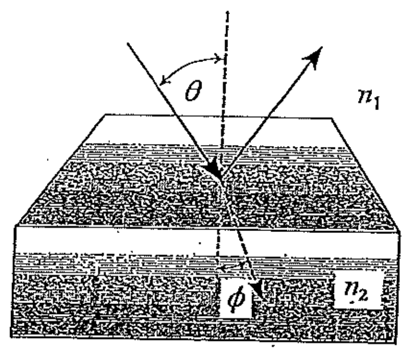
b) Show that the orbit $\vec{r} = \vec{r}(t)$ develops in a plane perpendicular to the constant $\vec{L}$ vector.

c) Show that $\vec{A} \cdot \vec{L} = 0$, hence, $\vec{A}$ also lies in the plane of the orbit.

d) Obtain the equation of the orbit by developing $\vec{A} \cdot \vec{r} = Ar\cos\theta$. Recognize that that is the equation of a conic section and identify its eccentricity ε in terms of $|\vec{A}| = A$. In which direction does $\vec{A}$ point, relative to the orientation of the conic section?

e) For which values of A do we have circular, elliptic, parabolic, and hyperbolic orbits?

Consider a plane electromagnetic wave incident at angle θ on a plane boundary separating two non-conducting media with indices of refraction n_1 and n_2, as shown. The space and time dependences of the incident, reflected, and transmitted waves have the forms: $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, $e^{i(\vec{k}'\cdot\vec{r}-\omega t)}$, and $e^{i(\vec{k}''\cdot\vec{r}-\omega t)}$, respectively.



(a) Write the expressions for the electric and magnetic fields of the incident wave.

(b) For the case in which the magnetic vector of the incident wave is parallel to the boundary (polarization in the plane of incidence) show that the ratio of the reflected to the incident amplitude of the wave, r_p, is as follows:

$$r_p = \frac{-n\cos\theta + \cos\phi}{n\cos\theta + \cos\phi}, \text{ where } n = n_2/n_1.$$

HINT: Apply the boundary conditions.

(c) Using Snell's law, show that the expression for r_p can be rewritten as:

$$r_p = \frac{-n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}}{n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}}.$$

(d) Consider the case for which $n < 1$. In this case show that, when the angle of incidence

$$\theta > \sin^{-1} n,$$

the square of the amplitude of reflection coefficient r_p becomes unity. Describe what this means physically.

Consider a low-density plasma that consists of free electrons of mass m and charge $q = -e$. There are N charges per unit volume. Assume that the density is uniform and interactions between the charges can be neglected [Imagine that there is a background of fixed charges $q = +e$ so that the plasma is overall neutral. Such a background could be provided by ions with masses $m_i >> m$, hence their contribution to the conductivity would be negligible]. Plane electromagnetic waves (frequency ω, wave number k) are incident on the plasma.

a) Starting with the equation of motion for a single electron, determine the total current density carried by the charges.

b) Using the result from (a), determine the conductivity of the plasma, σ, as a function of ω.

c) Starting from Maxwell's equations, determine the dispersion relation for the plasma. Note that it will be a function of ω. [alternatively, you could start with the wave equation, keeping in mind the non-zero conductivity of the plasma.]

d) What is the plasma frequency, ω_p, in terms of the parameters given above.

e) Determine the index of refraction of the plasma (write your answer in terms of the plasma frequency).

f) Describe what happens to the propagation of waves in the plasma when $\omega < \omega_p$.

Consider a surface with N_0 absorption sites, in equilibrium with an ideal gas. Each site can absorb at most one molecule from the gas, binding it with an energy $(-\varepsilon_0) < 0$.

a) Assuming that N molecules are absorbed on the surface, determine the canonical partition function $Q_N(\beta)$ for the corresponding surface system.

Hint: First show that the single $E = -N\varepsilon_0$ energy level has a degeneracy $g(N) = \dfrac{N_0!}{N!(N_0 - N)!}$.

b) Determine the grand-canonical partition function $Z(z,\beta) = \sum_{N=0}^{N_0} z^N Q_N(\beta)$ for the surface system, in terms of the surface fugacity $z = \exp(\beta\mu)$, where μ is the chemical potential and $\beta = 1/kT$.

c) Determine the average number $\overline{N} = z\dfrac{\partial}{\partial z}(\log Z)_\beta$ of molecules absorbed on the surface, and the corresponding covering ratio $\theta \equiv \dfrac{\overline{N}}{N_0}$, in terms of z.

d) Determine the grand-canonical partition function $Z_g(z_g,\beta)$ for the ideal gas, in terms of the gas fugacity z_g.

e) Determine the average number $\overline{N}_g$ of molecules in the gas, in terms of z_g.

f) What is the condition of equilibrium between the gas and the surface system? From that, derive the Langmuir isotherm $\theta = \dfrac{P}{P + f(T)}$, where P is the gas pressure and $f(T)$ is a function of the absolute temperature, which you must determine.

For a Classical Ideal Gas at a temperature T, provide the (Maxwell)

a) Probability $P(v)dv$ that any one molecule has a speed between v and $v + dv$. Include the proper normalization.

b) Determine the most probable speed v^*, average speed $\langle v \rangle$, and root-mean-square speed $v_{rms} = \sqrt{\langle v^2 \rangle}$. Also determine the standard deviation $\sigma = \sqrt{\langle v^2 \rangle - \langle v \rangle^2}$.

c) For a given $T = T_1$, plot $P(v)$ as a function of v. Report v^*, $\langle v \rangle$, and v_{rms} on the graph. How does that plot scale with increasing T? Plot $P(v)$ again for another $T = T_2 > T_1$.

d) At which high temperature T_c might there be roughly a 2.5% fraction of molecules that would have speeds greater than the speed of light? Consider that for a molecule in H_2 gas, $mc^2 \cong 1{,}876\ MeV$, while $k_B T_0 \cong \frac{1}{40} eV$ at $T_0 = 290$ K.

Positronium is a hydrogen-like system consisting of a bound state of an electron and a positron (a positively charged electron; both the electron and positron are spin-½ particles). The positronium ground state consists of a singlet and three triplet sub-states. The triplet states are degenerate and their energies are ΔE higher than that of the more stable singlet state. The splitting between these states is due to a spin-spin interaction of the form:

$$H_0 = \frac{A}{\hbar^2}\vec{S}_1 \cdot \vec{S}_2 \ ,$$

where A is a positive constant.

a) Determine the four eigenstates $|s,m\rangle$ of the total spin S^2 and its S_z component as linear combinations of the four $|m_1,m_2\rangle$ eigenstates of S_{1z} and S_{2z} of the electron and positron, respectively.

b) Use four orthonormal spinors

$$\chi_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \chi_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad \chi_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \chi_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

to represent the $|s,m\rangle$ eigenstates and find H_0 in this representation.

c) Find ΔE in terms of A.

Assume now that a magnetic field $\vec{B}$ is applied along the z-direction. The total Hamiltonian now is $H = H_0 + H'$, where H' is the contribution due to the magnetic field.

d) Show that $H' = \frac{b}{\hbar}\left(S_z^{(1)} - S_z^{(2)}\right)$ and determine b.

e) Find H' in matrix form in the $|s,m\rangle$ representation of part (b).

f) Find the exact energy eigenvalues of the total Hamiltonian H. Determine which eigenvalues of H_0 are split by H' and which eigenvalues of H_0 remain degenerate and explain why.

g) Plot the energy eigenvalues of the total H as a function of the magnetic field magnitude B from zero to infinity and discuss their behavior in the limits of $b << A$ and $b >> A$.

In the Schrodinger Picture (SP), States $|\psi_S(t)\rangle$ evolve in time according to the Schrodinger Equation (SE) and its formal solution, i.e.,

$$i\frac{d}{dt}|\psi_S(t)\rangle = H, \qquad |\psi_S(t)\rangle = e^{-iHt}|\psi_S(0)\rangle.$$

Here we assumed for simplicity a time-independent Hamiltonian H and units for which $\hbar = 1$. In the SP, Observables $R_S(t)$ may only depend on time explicitly. A special Observable is the State Operator $\rho_S(t)$. Let us simply consider a Pure State, for which

$$\rho_S(t) = |\psi_S(t)\rangle\langle\psi_S(t)|.$$

a) Determine the Time Evolution of $\rho_S(t)$ in terms of $\rho_S(0)$ at $t = 0$.

b) Determine the Equation of Motion for $i\frac{d}{dt}\rho_S(t)$, corresponding to the SE.

Let us further assume that

$$H = H_0 + H_1, \text{ with } [H_0, H_1] \neq 0.$$

The Interaction Picture is defined by the Transformations

$$|\psi_I(t)\rangle = e^{iH_0t}|\psi_S(t)\rangle, \qquad R_I(t) = e^{iH_0t}R_S(t)e^{-iH_0t}.$$

c) Determine the Equation of Motion for $i\frac{d}{dt}|\psi_I(t)\rangle$ in terms of $H_{1I}(t)$.

d) Determine the Equation of Motion for $i\frac{d}{dt}R_I(t)$ in terms of H_0 and including $\left(\frac{\partial R_S(t)}{\partial t}\right)_I$, besides $R_I(t)$.

e) Determine the Equation of Motion for $i\frac{d}{dt}\rho_I(t)$ in terms of $H_{1I}(t)$.

Consider a Particle in an External Potential, for which $H_0 = \frac{1}{2M}P_s^2$, $H_1 = V(Q_S)$.

f) Determine $Q_I(t)$ and $P_I(t)$ in terms of Q_S and P_S.

Consider the Hamiltonian of the H-atom in atomic units,

$$H = \frac{1}{2}P^2 - \frac{e^2}{r} \ .$$

That Hamiltonian conserves the Lenz Vector $\vec{A} = \frac{1}{r}\vec{Q} + \frac{1}{2}\left[\vec{L}\times\vec{P} - \vec{P}\times\vec{L}\right]$.

a) Show that $\vec{L}\cdot\vec{P} = \vec{P}\cdot\vec{L} = 0$, or $\vec{L}\cdot\vec{Q} = \vec{Q}\cdot\vec{L} = 0$, since $\vec{L} = \vec{Q}\times\vec{P}$.

Further consider (without proving them) the following relations,

$$[A_i, A_j] = -2i\varepsilon_{ijk}HL_k, \qquad [L_i, A_j] = i\varepsilon_{ijk}A_k \quad ,$$

$$\vec{A}\bullet\vec{A} = I + 2H(L^2 + I), \qquad \vec{L}\cdot\vec{A} = \vec{A}\cdot\vec{L} = 0 \ ,$$

where I is the identity operator.

Consider bound states with discrete energy eigenvalues E<0, and define in the subspace of each one, where $H \rightarrow$ E, the operators

$$\vec{W} = (-2E)^{-\frac{1}{2}}\vec{A}, \qquad \vec{J}_{(1)} = \frac{1}{2}\left(\vec{L} + \vec{W}\right), \qquad \vec{J}_{(2)} = \frac{1}{2}(\vec{L} - \vec{W}).$$

b) Show that $[J_{(\alpha)i} \ , \ J_{(\beta)j}] = i\varepsilon_{ijk}J_{(\alpha)k}\,\delta_{\alpha\beta}$ and $J_{(1)}{}^2 = J_{(2)}{}^2$, for $\alpha, \beta = 1, 2$. Then, what are the possible eigenvalues of $J_\alpha{}^2$?

c) In terms of those, what are the possible eigenvalues of L^2, for $\vec{L} = \vec{J}_{(1)} + \vec{J}_{(2)}$, within the subspace of a given $E < 0$? What is the dimension of that vector space?

d) From all the previous relations and results, determine the spectrum of the bound-state eigenvalues, $E_n < 0$, of H, and their degeneracies. Show explicitly the relations between $J_{(\alpha)} = J$ and the "principal quantum number" n and the degeneracy of the E_n-states.