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THE CATHOLIC UNIVERSITY OF AMERICA

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**Ph.D. Comprehensive Examination**

**Physics Department**

**Spring 2008**

**Thursday, March 27, and Friday, March 28, 2008**

## Room 133 - Hannan Hall

**GENERAL INSTRUCTIONS:**

This examination is divided into four sections as follows:

**Thursday, March 27, 2008**

| | |
|---|---|
| 9:00 a.m. - 12:00 Noon | Classical Mechanics - 2 questions |
| 1:00 P.M. - 5:00 P.M. | E & M - 3 questions |

**Friday, March 28, 2008**

| | |
|---|---|
| 9:00 a.m. - 12:00 Noon | Stat Mech. - 2 questions |
| 1:00 P.M. - 5:00 P.M. | Quantum Mech. - 3 questions |

***DO EACH PROBLEM IN A SEPARATE BLUE BOOK***

**PUT YOUR NAME ON EACH BOOK**

**LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS**
**FOR EXAMPLE: Mechanics 600-1**

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## RULES FOR THE PHYSICS COMPREHENSIVE EXAMINATION

**Spring 2008**

In order to assure an equitable and fair Comprehensive Examination for all students the following rules will apply to the examination administered by the Department of Physics.

1. The examination will be closed book. Students will not be permitted to bring to the examination room materials such as notes, reference books, calculators, or other aids of any form. The Physics Department will supply copies of the *CRC Mathematical Handbook,* Schaum's *Mathematical Handbook, Table of Functions* by Jahnke and Emde, and the NBS *Handbook of Mathematical Functions*, for the use of those taking the exam should they feel the need for these references during the examination.

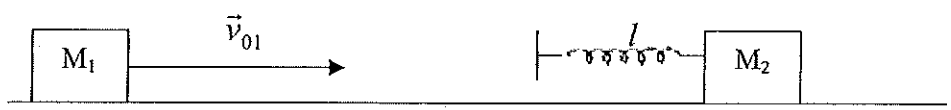
   The Physics Department will supply calculators for use during the examination.

   Students who want to use their own mathematical tables and/or handbooks must get permission from the Chairman prior to the beginning of the comprehensive examination. These tables and/or handbooks will remain under the control of the department until after the examination is completed, but will be available to their owners during the exam.

2. If needed, students will be allowed a short break, not to exceed ten minutes, during the examination period. While absent from the examination room students may not consult material or other sources of information concerning any matter pertaining to the examination. Students who leave the examination room will turn in all of their examination materials to the proctor before leaving the room. Examination papers will be returned to the students when they reenter the examination room. Students who fail to return within ten minutes will be considered to have completed that portion of the examination.

   **Only one student will be permitted to leave the examination room at a time.**

A mass $M_1$ is sliding on a frictionless plane, with velocity $\vec{v}_{01}$ initially, until it makes contact at $t = 0$ with a massless spring of force constant $K$ and rest length $l$. The other side of the spring is attached to a mass $M_2$, initially at rest. The spring thus contracts up to a minimum length $x_{min}$, and then rebounds until $M_1$ separates again from it. This provides a basic model for a one-dimensional elastic collision between $M_1$ and $M_2$. See Figure.




(a) Determine the equations of motion for the coordinates $x_1$ and $x_2$ of the two masses during the collision process.

(b) Introduce the relative coordinate $x = x_2 - x_1$ and the Center-of-Mass coordinate $X_{CM}$. Determine the equations of motion for $x$ and $X_{CM}$, and solve them. Determine $x_{min}$, in particular.

(c) Determine the final velocities $\vec{v}_1$ and $\vec{v}_2$ when the $M_1$ mass separates again from the spring.

(d) Now consider this elastic collision most generally and independently of the spring model. From conservation of momentum and energy initially and finally, determine again $\vec{v}_1$ and $\vec{v}_2$ and compare them with the spring-model result obtained in (c).

(e) Determine $\vec{v}_1$ and $\vec{v}_2$ in the limiting cases $M_1 = M_2$, $M_1 >> M_2$, and $M_1 << M_2$, and interpret physically these results.

Consider the central-force problem for an attractive potential of the form $V(r) = -\frac{k}{r^n}$, where $k$ and $n$ are two given real positive numbers.

(a) Recalling the derivation of the equivalent one-dimensional problem, sketch the effective potential $W(r) = \frac{L^2}{2\mu r^2} - \frac{k}{r^n}$ as a function of $r$ for two typical cases with $0 < n < 2$ and $n > 2$. What do $L$ and $\frac{L^2}{2\mu r^2}$ represent physically?

(b) Write the effective force equation corresponding to $\mu\ddot{r}$. For what values of $r_0$ and $E_0$ energy are circular orbits possible?

(c) What is the period $T_0$ or the angular frequency $\omega_0 = \frac{2\pi}{T_0}$ of revolution for circular orbits, in terms of $r_0$? Compare your result with Kepler's Third Law for the $n = 1$ Newtonian case.

(d) Now consider a small perturbation from a circular orbit, inducing a small energy change from $E_0$. In which cases do the perturbed orbits remain stable and nearly circular? What happens in the other cases?

(e) Expand $r \cong r_0 + \eta$ in the effective force equation from stable nearly circular orbits, and obtain the frequency $\Omega$ for small oscillations of $r$ near $r_0$, in terms of $r_0$.

(a) Starting with Maxwell's Equations, derive the equation for constancy of charge:

$$\nabla \cdot \vec{J} + \frac{\partial}{\partial t}\rho = 0$$

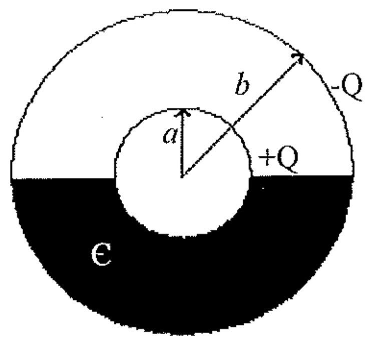
(b) Now assume the case in vacuum. Derive the vector wave equations for $\vec{E}$ and $\vec{B}$.

(c) Noting that solutions of the wave equation are of the form:
$\vec{E} = \bar{a}_1 E_0 e^{i(k\cdot r-\omega t)}$, $\vec{B} = \bar{a}_2 B_0 e^{i(k\cdot r-\omega t)}$, where $a_1$ and $a_2$ are unit vectors, show that: the unit vectors are orthogonal, $E_0=cB_o$, and the waves propagate perpendicular to the plane formed by the unit vectors with a phase velocity $v_{ph} = c$.

(d) Now, consider wave propagation in a conducting medium, so that the current density is related to the electric field by Ohm's law: $\vec{J} = \sigma\vec{E}$. Show that, without the displacement current term $\frac{\partial E}{\partial t}$, the continuity equation presents unacceptable constraints on the sources of the field.

(e) Show that the same condition described in part d) precludes the existence of waves (i.e., attempt to derive the wave equation with a non-zero current $\vec{J}$, but without a displacement current).

Two concentric conducting spheres of inner and outer radii, $a$ and $b$, respectively, carry charges $\pm Q$. The empty space between the spheres is half-filled by a hemispherical shell of dielectric (of dielectric constant $\epsilon$), as shown:




(a) Find the electric field everywhere between the surfaces $(a \leq r \leq b)$. Assume a potential of form:

$$\Phi(r,\theta)=\sum_{l=0}^{\infty}\left[A_l r^{l}+B_l r^{-(l+1)}\right]P_l(\cos\theta)$$

(b) Calculate the surface-charge distribution on the inner sphere.

(c) Calculate the polarization charge density on the surface of the dielectric at $r = a$.

Consider a long straight wire of negligible cross-sectional area $\delta$A and uniform linear charge density $\lambda'$. The wire lies at rest along the $z'$-axis of an inertial frame K$'$, and moves with constant velocity $\vec{v}$ along the z-axis of an inertial laboratory frame K.

(a) Determine the 4-vector potential $\mathrm{A}^{\mu'}(x') = (\phi'(x'), \vec{\mathrm{A}}'(x'))$ in the rest-frame K$'$ of the wire. Here $x' = x^{\alpha'} = (ct', \vec{x}')$ denote the space-time coordinates of an event in K$'$.
Hint: Determine the electric field $\vec{\mathrm{E}}'(\rho')$ first, exploiting cylindrical symmetry and Gauss Theorem in cylindrical coordinates $(\rho', \varphi', z')$. Choose $\vec{\mathrm{A}}'(x') = 0$.

(b) Consider the Lorentz Transformation $ct = \gamma(ct' + \beta z')$, $z = \gamma(z' + \beta ct')$, $\rho = \rho'$ to the laboratory frame K. Determine the corresponding $\mathrm{A}^{\mu}(x)$ in K, and the resulting electric and magnetic fields $\vec{\mathrm{E}}(x)$ and $\vec{\mathrm{B}}(x)$ in K.

(c) Determine the 4-current $J^{\mu}\delta A = (c\lambda, \vec{J}\delta A)$ in K from the Lorentz Transformation of the 4-current $J^{\mu'}\delta A = (c\lambda', \vec{0})$ in K$'$. Interpret the results in terms of the Lorentz contraction of the wire in K.

(d) Using $J^{\mu}\delta A$ as derived in K, determine $\vec{\mathrm{E}}$ and $\vec{\mathrm{B}}$ directly in K, using Gauss and Ampere Laws in K, where the wire is rigidly moving with velocity $\vec{v}$.

Consider a system of $N$ spin-½ magnetic moments $\vec{\mu}$ in thermal equilibrium at a temperature $T$ in a static magnetic field $\vec{B}$.

(a) Determine the canonical probability $p_+$ that any given $\vec{\mu}$ is aligned with $\vec{B}$, hence, in the ground state with energy $(-\varepsilon) = -\mu_B B$. Determine the alternate probability $p_-$ that $\vec{\mu}$ is anti-aligned with $\vec{B}$, hence, in the excited state with energy $\varepsilon = \mu_B B$, at given $T$.

(b) Show that the Energy-Temperature relation, or equation of state, is

$$E = -(N\varepsilon)\tanh(\beta\varepsilon), \text{ where } \beta = \frac{1}{KT}.$$

(c) Now suppose that $T < 0$, corresponding to a "population inversion." Why and how such an equilibrium state at a negative temperature may be achieved for this system?

Now that we have the N spin-½ system at an initial negative temperature $T_i < 0$, we put it in thermal contact with a small "thermometer" at an initial temperature $t_i > 0$. Suppose that the "thermometer" is an "ordinary" system, made for instance with an ideal gas of $n$ molecules, obeying the equations of state $u = \frac{3}{2}nKT$ and $PV = nKT$. After thermal contact is established, the N spin-½ system and the small thermometer reach a common final equilibrium temperature $T_f$.

(d) Determine the relation between the final $T_f$ and the initial $T_i < 0$, assuming $N >> n$. Consider in particular the limiting cases $|KT_i| << \varepsilon$ and $|KT_i| >> \varepsilon$. Why is the final $T_f$ always positive, with $KT_f >> \varepsilon$?

Hint: Use conservation of internal energy for the composite system before and after thermal contact is established between its two components.

Consider an ideal non-relativistic gas of $N$ particles in a given volume $V$, in the classical limit.

(a) Determine the canonical probability $p(\varepsilon, T)\,d\varepsilon$ that any one particle has an energy between $\varepsilon$ and $\varepsilon + d\varepsilon$, assuming that the gas system is kept at a given temperature $T$. Hint: Remember that the degeneracy is $g(\varepsilon)d\varepsilon = A\sqrt{\varepsilon}\,d\varepsilon$, and that $p(\varepsilon, T)\,d\varepsilon$ must ultimately integrate to 1.

(b) The corresponding microcanonical probability is $p(\varepsilon, E)\,d\varepsilon = \dfrac{\Omega(N-1, E-\varepsilon)}{\Omega(N,E)} \bullet g(\varepsilon)d\varepsilon$, assuming that the gas system is kept at a given energy $E$, i.e. thermally isolated. Explain the physical meaning of all three factors comprised in the expression of $p(\varepsilon, E)$ and its physical interpretation.

(c) Recall that $\Omega\,(N, E)\ \alpha\ E^{\frac{3N}{2}-1}$ microcanonically. Then consider $\log\{p(\varepsilon, E)\}$ and expand it for $N>>1$ and $E>>\varepsilon$, without concern for additive constants independent of $E$ and $\varepsilon$. Exponentiate the result to obtain $p(\varepsilon, E)$ and fix a multiplicative constant by integrating $p(\varepsilon, E)\,d\varepsilon$ to 1.

(d) For which relation between $E$ and $T$ does the microcanonical $p(\varepsilon, E)$ coincide with the canonical $p(\varepsilon, T)$? Which Equations of State does that $E$-$T$ relation represent in the microcanonical and canonical ensembles?

Useful Formulae:

$\Gamma(n) = (n-1)! = \int_0^\infty dx e^{-x} x^{n-1}$, $\mathrm{Re}(n) > 0$.

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

$\Gamma(n+1) = n\Gamma(n)$.

For a one-dimensional Harmonic Oscillator, we have the following relations

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right),\ a = \sqrt{\frac{M\omega}{2\hbar}}Q + i\frac{1}{\sqrt{2M\hbar\omega}}P,\ \ a\,|\,0\rangle = 0,\ a^\dagger\,|\,n\rangle = \sqrt{n+1}\,|\,n+1\rangle.$$

(a) Name and explain the meaning of these relations and symbols.

(b) Project/express $a\,|\,0\rangle = 0$ in momentum or $p$-representation, thus obtaining a first-order differential equation for the eigen-function $\psi_0(p) =< p\,|\,0>$ for the Ground State of $H$ in the $p$-rep.
Hint: $Q \to i\hbar\frac{d}{dp}$ and $\frac{d}{dp}\psi_0(p) = -g(p)\psi_0(p)$ for an appropriate function $g(p)$ that you must determine.

(c) Solve and obtain the normalized $\psi_0(p)$ explicitly.

(d) Derive the relation that generates recursively from $\psi_0(p)$ any Excited State eigen-function $\psi_n(p)$ of $H$ in the $p$-rep.

(e) Determine explicitly $\psi_1(p)$ and $\psi_2(p)$. What are their corresponding eigen-values for $H$?

In the Schroedinger Picture (SP), States $|\psi(t)\rangle$ obey the Equation of Motion, or Schroedinger Equation (SE),

$$i\frac{d}{dt}|\psi_S(t)\rangle = H|\psi_S(t)\rangle,$$

where we assume for simplicity a time-independent Hamiltonian $H$ and units for which $\hbar = 1$..

(a) Integrate formally the SE to express the Time Evolution of $|\psi_S(t)\rangle$, starting from $|\psi_S(0)\rangle$ at $t = 0$.

In the SP, Observables $R_S(t)$ may only depend on time explicitly. A special Observable is the State Operator $\rho_S(t)$. Let us simply consider a Pure State, for which

$$\rho_S(t) = |\psi_S(t)\rangle\langle\psi_S(t)|.$$

(b) Determine the Time Evolution of $\rho_S(t)$ in terms of $\rho_S(0)$ at $t = 0$.

(c) Determine the Equation of Motion for $i\frac{d}{dt}\rho_S(t)$, corresponding to the SE.

The Heisenberg Picture (HP) is defined by the Transformations

$$|\psi_H\rangle = e^{iHt}|\psi_S(t)\rangle,\ R_H(t) = e^{iHt}R_S(t)e^{-iHt}.$$

(d) Determine the Heisenberg Equation of Motion for $i\frac{d}{dt}R_H(t)$.

(e) Determine the State Operator $\rho_H$ in the HP and verify that $i\frac{d}{dt}\rho_H = 0$, using (d) and (c).

Consider a Harmonic Oscillator, for which

$$H = \frac{1}{2M}P_S^2 + \frac{1}{2}M\omega^2 Q_S^2.$$

(f) Determine the Heisenberg Equations of Motion for $i\frac{d}{dt}Q_H(t)$ and $i\frac{d}{dt}P_H(t)$.

(g) Solve those Equations to determine $Q_H(t)$ and $P_H(t)$ in terms of $Q_S$ and $P_S$.

Consider a Spin L=1 system prepared in a normalized eigenstate $|1\rangle_z$ of $L_z$ with maximum eigenvalue M=1, in units of $\hbar$.

(a) Either from simple symmetry considerations or explicit calculations determine the Probabilities $|\alpha_+|^2, |\alpha_0|^2, |\alpha_-|^2$ that this preparation yields values (+1), 0, (-1) for a measurement of the $L_x$ Spin component.

(b) Determine the average values ${}_z\langle 1 | L_x | 1\rangle_z$ and ${}_z\langle 1 | L_y | 1\rangle_z$.

(c) Determine the average values ${}_z\langle 1 | L_x^{\ 2} | 1\rangle_z$ and ${}_z\langle 1 | L_y^{\ 2} | 1\rangle_z$, and verify their consistency with ${}_z\langle 1 | L^2 | 1\rangle_z$. How does this result differ from the Classical Mechanics expectation and why?