GENERAL INSTRUCTIONS:

This examination is divided into four sections as follows:

**Thursday, March 16, 2006**
9:00 a.m. - 12:00 Noon  Classical Mechanics - 2 questions
1:00 P.M. - 5:00 P.M.  E & M - 2 questions

**Friday, March 17, 2006**
9:00 a.m. - 12:00 Noon  Thermodynamics/Stat. Mech. - 2 questions
1:00 P.M. - 5:00 P.M.  Modern Physics/Quantum Mech. - 2 questions

**DO EACH PROBLEM IN A SEPARATE BLUE BOOK**

**PUT YOUR NAME ON EACH BOOK**

**LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS**
FOR EXAMPLE: Mechanics 600-1 or Electromagnetism 500-2

**YOU MUST DO TWO (AND ONLY TWO) OF THE QUESTIONS IN EACH AREA.**

**OF THE TOTAL OF EIGHT QUESTIONS THAT ARE DONE, A MINIMUM OF TWO MUST BE FROM THE 600-LEVEL QUESTIONS.**
RULES FOR THE PHYSICS COMPREHENSIVE EXAMINATION

SPRING 2006

In order to assure an equitable and fair Comprehensive Examination for all students the following rules will apply to the examination administered by the Department of Physics.

1. The examination will be closed book. Students will not be permitted to bring to the examination room materials such as notes, reference books, calculators, or other aids of any form. The Physics Department will supply copies of the CRC Mathematical Handbook, Schaum's Mathematical Handbook, Table of Functions by Jahnke and Emde, and the NBS Handbook of Mathematical Functions, for the use of those taking the exam should they feel the need for these references during the examination.

The Physics Department will supply calculators for use during the examination.

Students who want to use their own mathematical tables and/or handbooks must get permission from the Chairman prior to the beginning of the comprehensive examination. These tables and/or handbooks will remain under the control of the department until after the examination is completed, but will be available to their owners during the exam.

2. If needed, students will be allowed a short break, not to exceed ten minutes, during the examination period. While absent from the examination room students may not consult material or other sources of information concerning any matter pertaining to the examination. Students who leave the examination room will turn in all of their examination materials to the proctor before leaving the room. Examination papers will be returned to the students when they reenter the examination room. Students who fail to return within ten minutes will be considered to have completed that portion of the examination.

Only one student will be permitted to leave the examination room at a time.
PhD/MS/Preliminary Comprehensive Exam

Spring 2006

Proctoring Schedule

Room 136 Hannan Hall

Thurs., March 16, 2006
9:00-10:30 am - Biprodas Dutta
10:30-12:00 noon - Leon Ofman
1:00-3:00 pm - Franz Klein
3:00-5:00 pm - Lorenzo Resca

Fri., March 17, 2006
9:00-10:30 - Charles Montrose
10:30-12:00 noon - Dan Sober
1:00-3:00 pm - Ian Pegg
3:00-5:00 pm - Pete Macedo
THE CATHOLIC UNIVERSITY OF AMERICA

Department of Physics
200 Hannon Hall
Washington, D.C. 20064
202-319-5315
Fax 202-319-4448

Ph.D. Comprehensive Examination

Physics Department

Spring 2006

Thursday, March 16, and Friday, March 17, 2006

Room 136 - Hannan Hall

GENERAL INSTRUCTIONS:

This examination is divided into four sections as follows:

Thursday, March 16, 2006
9:00 a.m. - 12:00 Noon  Classical Mechanics - 2 questions
1:00 P.M. - 5:00 P.M.  E & M - 3 questions

Friday, March 17, 2006
9:00 a.m. - 12:00 Noon  Stat Mech.  - 2 questions
1:00 P.M. - 5:00 P.M.  Quantum Mech. - 3 questions

DO EACH PROBLEM IN A SEPARATE BLUE BOOK

PUT YOUR NAME ON EACH BOOK

LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS
FOR EXAMPLE: Mechanics 600-1
**THE CATHOLIC UNIVERSITY OF AMERICA**

*Department of Physics*
200 Hannon Hall
Washington, D.C. 20064
202-319-5315
Fax 202-319-4448

PhD Oral Comprehensive Exam
**SPRING 2006**
Room 203 or 231 - Conference Rooms (whichever is available)

<table>
<thead>
<tr>
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<th>STUDENT</th>
<th>COMMITTEE</th>
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<td>Tuesday, March 21, 2006</td>
<td>203 or 231</td>
<td>Marcio Melendez</td>
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<td>Biprodas Dutta</td>
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PhD/MS/PRELIB. Comprehensive Examination

Spring 2006

Thursday, March 16, and Friday, March 17, 2006

THE WRITTEN EXAM WILL BE HELD IN ROOM

136 HANNAN HALL

THE EXAM WILL BE FROM 9:00 AM - 5:00 PM ON THURSDAY March 16, AND 9:00 AM - 5:00 PM ON FRIDAY, March 17, 2006.
THERE WILL BE A GET TOGETHER ON THE 2ND FLOOR AT 5:00 PM ON FRIDAY.

THE ORAL PORTION OF THE PhD COMPREHENSIVE EXAM WILL BEGIN THE FOLLOWING WEEK.

PLEASE CALL (202) 319-5315 OR STOP BY THE PHYSICS DEPARTMENT OFFICE ON FRIDAY TO INQUIRE ABOUT YOUR SCHEDULED ORAL EXAM.
A "loop-the-loop" track consists of an entry section (a-b), a circular section of radius R (b-c-b), and an exit section (d-d), as shown in the figure.

a) A particle of mass \( m_1 \) starts from rest at point a and slides without friction along the track. What is the minimum vertical height of the starting point in order for the particle to stay in contact with the track and make it around the loop?

b) The particle is now replaced by a cylinder of radius \( r \), length \( d \), and mass \( m_2 \). The cylinder rolls without slipping along the track. What is the minimum vertical height of the starting point in order for the cylinder to stay in contact with the track and make it around the loop? (You may assume that \( R \gg r \).)

c) Compare your results in parts (a) and (b) and explain any difference between them.

d) Find the linear velocity at the bottom (point b) for the particle and for the cylinder. Explain any difference between these results.

e) Assume now that the ramp from point a to b is a straight line that makes an angle \( \theta \) with the horizontal. Consider the motion of the particle when a friction force acts which is linearly proportional to the speed (\( F = \alpha v \), where \( \alpha \) is a constant and \( v \) is the speed). Find the height of the starting point (point a) such that the particle has the same speed at point b as in the case without friction (part (a)).
Consider a simple un-damped harmonic oscillator composed of an ideal mass-spring system with mass $m$, spring constant $k$, and natural frequency $\omega$. At time $t = 0$, the mass is given an initial displacement $x_0$ and an initial velocity $v_0$.

a) Write out the equation of motion for this system.

b) Show that the displacement at time $t$ is given by

$$x = x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t.$$ 

Consider now the same system but initially at rest (i.e., with zero initial displacement and zero initial velocity). The mass is now subjected to a "square-pulse" external force $F(t)$ where:

$$F(t) = \begin{cases} 
0 & \text{for } 0 < t < t_1 \\
F_0 & \text{for } t_1 < t < t_2 \\
0 & \text{for } t > t_2.
\end{cases}$$

We wish to find the effect of this force by considering it to be composed of a series of impulses each of duration $dt$, as follows.

c) Find the displacement $dx$ caused by the impulse $F_0 dt$ that occurs at time $t_1$. Proceed by applying your result from part (b) and treating the velocity caused by the impulse as the "initial" velocity.

d) From your result from part (c), find the total displacement caused by $F(t)$.

e) If now we return to our original system with non-zero initial displacement $x_0$ and non-zero initial velocity $v_0$ and subject it to the same force $F(t)$, what is the total displacement due to the initial conditions and $F(t)$?
A total charge $q$ is distributed on the surface of a disc of radius $R$ as follows: The disc is divided into eight equal "wedge" shaped segments, as shown in the figure. The shaded regions are of equal uniform charge density, whereas the unshaded regions are uncharged.

a) Find the electrostatic potential on the axis of the disc (i.e., along the $z$-direction).

b) Show that at large distances the potential function behaves as that of a point charge.
Consider a uniformly charged sphere of charge \( Q \) and radius \( R \) rotating about the z-axis at a constant frequency of rotation \( \omega \).
Due to spherical symmetry, the magnetic scalar potential inside and outside the sphere can be expanded in terms of Legendre polynomials:

\[
\Phi_{\text{out}} = \sum_{m=0}^{\infty} a_m r^m P_m(\cos \theta) \quad \text{and} \quad \Phi_{\text{in}} = \sum_{m=1}^{\infty} b_m r^m P_m(\cos \theta) \quad \text{with} \quad \overrightarrow{B} = -\nabla \Phi.
\]

The coefficients \( a_m \) and \( b_m \) are to be determined by applying boundary conditions as follows.

a) Show that Gauss' law applied to a small thin box at the surface of the sphere provides the boundary condition: \( (\nabla \Phi_{\text{out}} - \nabla \Phi_{\text{in}}) \cdot \hat{n} = 0 \).

b) Use the boundary condition given in a) to find a relation between \( a_m \) and \( b_m \).

c) Show that Ampere's law applied to a very thin rectangular loop at the surface of the sphere provides a 2nd boundary condition: \( (\nabla \Phi_{\text{out}} - \nabla \Phi_{\text{in}}) \cdot \hat{\theta} = \mu_0 j_z(\theta) \), where \( \hat{\theta} \) is the unit vector in direction of increasing \( \theta \).

d) Determine the surface current density \( j_z(\theta) \) at the angle \( \theta \) from the z-axis.

e) Use the 2nd boundary condition from c) and the result from d) to determine the magnetic potential \( \Phi_{\text{out}} \) and the magnetic induction \( \overrightarrow{B}_{\text{out}} \) outside the spinning sphere.

f) Show that the magnetic dipole moment of the spinning sphere is given by: \( \overrightarrow{m} = \frac{QR^2}{3} \overrightarrow{\omega} \).
Consider a long thin elastic rod (e.g., a section of a rubber band) with length $L$, cross-sectional area $A$, and volume $V=AL$. When the first and second laws of thermodynamics are applied to this system they yield:

$$dU = TdS - pdV + fdL,$$

where $p$ is the pressure and $f$ is the tension the rod exerts when its length is increased.

a) Define and provide physical interpretations of each of the four differential quantities involved in the above expression. In particular, explain all of their signs (positive or negative).

b) Which thermodynamic variables in the above equation are intensive and which are extensive?

c) What are the "natural" variables for this system in the $U$-representation?

d) Determine the $(T,p)$ Maxwell relation in the $U$-representation.

e) Through appropriate Legendre transformations, or otherwise, determine the thermodynamic potentials $H(S, p, L)$ and $G(T, p, L)$ and their corresponding differentials, $dH$ and $dG$.

f) Derive the Maxwell relations:

$$\left(\frac{\partial T}{\partial L}\right)_{T,p} = -\left(\frac{\partial f}{\partial S}\right)_{L,p} \quad \text{and} \quad -\left(\frac{\partial S}{\partial L}\right)_{T,p} = \left(\frac{\partial f}{\partial T}\right)_{L,p}.$$
A building is heated to a temperature $T$ by means of a heat pump that uses a river at $T_0$ as a source of heat. The heat pump has ideal performance and consumes a constant power $W$.

a) What is the rate at which heat is supplied by the heat pump?

b) The building loses heat to its surroundings at a rate $\alpha(T-T_0)$, where $\alpha$ is a constant. Show that the equilibrium temperature of the building, $T_e$, is given by

$$T_e = T_0 + \frac{W}{2\alpha} \left\{ 1 + \left( 1 + \frac{4\alpha T_0}{W} \right)^{1/2} \right\}.$$

c) What would be the equilibrium temperature if instead of using a heat pump, the constant power $W$ is used to supply a simple heater that is 100% efficient in converting the power to heat? Why is this less preferable?
When a beam of electrons is accelerated through an electrical potential difference $V$ and impinges on a tungsten anode, x-rays are emitted. The figure shows the x-ray spectrum for two different accelerating voltages; $I(\lambda)$ is the x-ray intensity emitted in the wavelength range $d\lambda$

a) What is the physical origin of the broad background in the x-ray spectrum?

b) The "cut-off" wavelength $\lambda_m$ decreases with increasing $V$ according to the empirical "Duane-Hunt rule"

$$\lambda_m = \frac{A}{V}$$

where the constant $A$ is found to be 1240 nm V. Provide a physical explanation for the Duane-Hunt rule and hence determine $A$ in terms of fundamental constants.

c) What is the physical origin of the sharp lines in the spectrum?

d) Why do the sharp lines appear in groups ("series")? How do the K- and L-series arise?

e) Why is there no K-series when $V=40$ kV?

f) Moseley found important relationships between the frequency of a particular line $f$ and the atomic number of the target element $Z$. In particular, for the first line in the $K$-series ($K_\alpha$), he found: $f = B(Z - 1)^2$.

(i) Explain why $f$ depends on $(Z - 1)$ rather than $Z$.

(ii) Use the Bohr model to derive this relationship.

![X-ray Spectrum Diagram](image-url)
A particle of mass $m$ moves in one dimension under the influence of a potential $V(x)$. Suppose it is in a non-degenerate energy eigenstate

$$
\psi(x) = \left( \frac{a^2}{\pi} \right)^{1/4} e^{-a^2x^2/2},
$$

with energy $E = \frac{\hbar^2 a^2}{2m}$.

a) Find the mean position of the particle $\langle x \rangle$.

b) Find $\langle x^2 \rangle$.

c) Find the mean momentum of the particle $\langle p \rangle$.

d) Find $\langle p^2 \rangle$.

e) Show that these results are consistent with the uncertainty principle.

f) Find $V(x)$.

g) Explain how you could find the probability $P(p)dp$ that the momentum of the particle is between $p$ and $p + dp$.

h) If now two spin-1/2 particles are placed in this potential and each has energy $E$, write out the possible two-particle eigenstates for the system.
A simple pendulum consists of a particle of mass $m$ attached to a massless rigid rod, fixed at the other end. Initially, the pendulum is at rest in the uniform gravitational field, in the position of stable equilibrium, such that the angle between the rod and the vertical is $\theta = 0$. Suddenly, the mass is given precisely the minimal kinetic energy that is required in order to swing all the way to the position of unstable equilibrium at the top (i.e., $\theta = \pi$). Starting from the equation of energy conservation, determine fully the time dependence of the angle $\theta$. In particular, how long does it take for the particle to reach the top, given this critical initial condition?
Suppose the Hamiltonian for the motion of a single particle is given by:

\[ H(x, p_x) = \frac{kx^2}{2} + \frac{p_x^2}{2m}. \]

a) Rewrite the Hamiltonian using normalized coordinates: \( p = \frac{p_x}{\sqrt{m}} \) and \( q = \sqrt{k}x \), and find Hamilton's equations of motion for \( H(p, q) \).

b) Consider the transformation: \( p = \sqrt{2P} \cos Q, \quad q = \sqrt{2P} \sin Q \). Use Poisson brackets to show that the transformation is canonical.

c) Show that Hamilton's equations of motions for \( H'(P, Q) \) are easily integrated and provide simple solutions for \( P(t) \) and \( Q(t) \). Find the solutions for \( p(t) \) and \( q(t) \) by substituting \( P(t) \) and \( Q(t) \) in the relations \( p(P, Q) \) and \( q(P, Q) \) from part b).

d) Find a function \( F_t(q, Q) \) that generates the transformation given in part b).
Consider a set-up of Oblique Incidence from a medium (A) of real index of refraction $n_A = \sqrt{\varepsilon_A} > n_B$ filling the $z < 0$ half-space into a medium (B) of real index of refraction $n_B < n_A$ filling the $z > 0$ half-space. Take $\vec{E}(x,t) = E_0 e^{i(k_1 x - \omega t)} \hat{\imath}_x$ as the Incident Electric Field, $\vec{E}''(x,t) = E_0 e^{i(k_2 x - \omega t)} \hat{\imath}_x$ as the Reflected Electric Field, and $\vec{E}'(x,t) = E_0 e^{i(k_1 x - \omega t)}$ as the Refracted Electric Field. Assume the condition $\sin \theta_1 > \sin \theta_0 = \frac{n_B}{n_A} = k'$ for the Incidence Angle $\theta_1$ and Snell's Law $k' \sin \theta_r = k \sin \theta_i$ for the complex Refraction Angle $\theta_r$. Correspondingly, $\vec{k}' = k' \hat{\imath}_r = \frac{\omega}{c} n_B \hat{\imath}'_r$ is a complex wave-vector, with $\hat{\imath}'_r = \hat{\imath}'_r + i \hat{\imath}'_t$ and $\hat{\imath}_r \cdot \hat{\imath}_r = 1$.

a) Show that $\vec{k}' = (k \sin \theta_i) \hat{\imath}_x + i \hat{\imath}_z$, and determine $\chi$.

You may proceed as follows. Consider at first the $\sin \theta_1 < \sin \theta_0$ condition, as shown in the Figure, and decompose trigonometrically $\vec{k}'$ in $\hat{\imath}_x$ and $\hat{\imath}_z$. Set $\cos \theta_r = \sin \theta_0 \sqrt{\left( \frac{1}{\sin \theta_r} \right)^2 - 1}$ and transform $\sin \theta_r$ using Snell's Law. Then allow the $\sin \theta_1 > \sin \theta_0$ condition, allowing the square root of a negative number to become imaginary.

b) From Faraday's Law $\vec{\nabla} \times \vec{E}' = -\frac{1}{c} \frac{\partial \vec{B}'}{\partial t} = -i \frac{\omega}{c} \vec{B}'$, determine the $\vec{B}'(x,t)$ field in Medium (B).

c) Determine $\vec{S}'(x) = \frac{c}{8\pi} \vec{E}' \times \vec{B}'$ in Medium (B), and the corresponding time-averaged Power Flux. Is power transmitted into Medium (B)?

d) Referring to the appropriate Fresnel Formula $\frac{E'_0}{E_0} = \frac{n_A \cos \theta_1 - \sqrt{n_B^2 - n_A^2 \sin^2 \theta_1}}{n_A \cos \theta_1 + \sqrt{n_B^2 - n_A^2 \sin^2 \theta_1}}$ confirm that $\sin \theta_1 > \sin \theta_0 = \frac{n_B}{n_A}$ provides a condition for Total Internal Reflection in Medium (A).
Below the critical magnetic field $H_c$, superconductors (of type I) exhibit the Meissner effect, in which all magnetic fields are excluded from the volume of the superconductor. Consider a superconducting sphere of radius $a$ placed in an otherwise uniform magnetic induction $B_0$. Due to the induced dipole field, the total magnetic induction outside the sphere is given by $B(r) = B_0 + \frac{\mu_0}{4\pi} \left( \frac{3(m \cdot \hat{r})\hat{r} - m}{r^3} \right)$.

a) Since the magnetic field is excluded from the sphere, the magnetic induction $B$ is tangent to the sphere at its surface. Use this boundary condition to determine the induced dipole moment $m$.

b) Show that the energy required to place the sphere in the magnetic field is $w = \frac{\pi a^3 B_0^2}{\mu_0}$.

*Hint:* The Maxwell stress tensor shows that the magnetic induction exerts a pressure on the surface of the sphere. Compute the work $\delta w$ done on the field when the radius of the sphere expands from $r$ to $r + \delta r$. Or equivalently: integrate the magnetic energy density over the volume of the sphere.

c) Suppose the superconducting sphere is placed in the non-uniform field above the end of a solenoid (see diagram). In the following, the $z$ axis is chosen to be vertical, along the axis of the solenoid (with $z = 0$ at the upper end of the solenoid), and $\rho = \sqrt{x^2 + y^2}$ is the radial distance from the $z$-axis. Provided that the radius $a$ of the sphere is sufficiently smaller than the radius $R$ of the solenoid, i.e., the magnetic field can be regarded as locally uniform, the energy of the sphere is then given by $w = \frac{\pi a^3 (B_z^2 + B_\rho^2)}{\mu_0}$, where $B_z$ is the field component along the axis of the solenoid and $B_\rho$ is the radial component:

$$B_\rho \approx \frac{\mu_0 J_\phi}{4} \frac{R^2 \rho}{(R^2 + z^2)^{3/2}}$$ and $B_z \approx \frac{\mu_0 J_\phi}{2} \left( 1 - \frac{z}{(R^2 + z^2)^{1/2}} - \frac{3}{4} \frac{R^2 z \rho^2}{(R^2 + z^2)^{3/2}} \right)$, where $J_\phi$ is the azimuthal current per unit length of the solenoid.

Show that the position of the sphere above the solenoid is stable against small horizontal displacements if the radius $R$ of the solenoid is more than $\sqrt{24}$ times larger than the height $z$ of the sphere above the solenoid.

*Hint:* You may neglect terms of order $\rho^4$ when calculating $B_z^2$. 
A particle of mass $m$ and charge $q$ is restrained by an ideal spring (massless with zero free length) with a spring constant $\eta$. A monochromatic plane wave with an electric field

$$\vec{E}(x,t) = \vec{E}_0 e^{i(kx - \omega t)}, \quad (k = \omega / c)$$

impinges on this system, with a wavelength much greater than any displacement of the particle.

(a) Find the angular distribution of the scattered power per unit solid angle.

**Hint:** In the radiation zone, the vector potential, the magnetic induction, the electric field, and the angular distribution of radiated power of a harmonically time-varying dipole moment $\vec{p}_0 e^{-i\omega t}$, are given by

$$\vec{A} = -ik\vec{p}_0 \frac{e^{i(kr - \omega t)}}{r}; \quad \vec{B} = ik\vec{n} \times \vec{A}; \quad \vec{E} = -\vec{n} \times \vec{B}; \quad \frac{dP}{d\Omega} = \frac{c}{8\pi} \text{Re}[r^2 \vec{n} \cdot (\vec{E} \times \vec{B}')]$$

(b) Integrate your result in (a) to find the total scattered power (in all directions), and show that the total scattered power is:

(i) proportional to the fourth power of the frequency if the particle is massless, i.e., $m = 0$, $\eta \neq 0$; and

(ii) frequency independent if the particle is unrestrained, i.e., $m \neq 0$, $\eta = 0$. 

Consider a system of \( N \) non-interacting spin-3/2 particles with magnetic moments \( \mu_z = -g\mu_B J \), where \( g = 2 \), \( \mu_B = \frac{e\hbar}{2me} \), \( J = -3/2, -1/2, 1/2, 3/2 \). These particles are placed in a uniform magnetic field \( \vec{B} \), which generates four energy levels, \( \varepsilon = -3\varepsilon_B, -\varepsilon_B, \varepsilon_B, 3\varepsilon_B \), where \( \varepsilon_B = \mu_B B \) for each particle. The system is maintained at a temperature \( T \).

a) What are the canonical probabilities \( p(\varepsilon) \) that any one particle is in each of the four \( \varepsilon \)-levels?

b) What is the average energy \( \langle \varepsilon \rangle \) of any one particle?

c) Determine all four \( p(\varepsilon) \) and \( \langle \varepsilon \rangle \) to first order in \( \beta = 1/kT \) for \( kT >> \varepsilon_B \) and check the consistency of your results in this limit.

d) Determine the specific heat \( C \) (per particle) for \( kT >> \varepsilon_B \). Why does \( C \) vanish with increasing \( T \)?

e) Determine all four \( p(\varepsilon) \) and \( \langle \varepsilon \rangle \) to first order in \( e^{-\beta B} \) for \( kT << \varepsilon_B \).

f) Determine the specific heat \( C \) for \( kT << \varepsilon_B \). Why does \( C \) vanish exponentially at low \( T \)?

g) Could a "population inversion" be realized for this system and why? If so, for which temperatures and how?
The Hamiltonian of a two-dimensional dipolar molecule of mass $m$, moment of inertia $I$, and dipole moment $\mu$ in an electric field $E$ is

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{p_z^2}{2I} - \mu E \cos \theta$$

where $\theta$ is the angle between $\vec{E}$ and $\vec{\mu}$.

a) Find the canonical partition function of a system of $N$ such classical dipolar molecules confined to an area $A$.

b) Obtain low-temperature and high-temperature approximations for the specific heat at constant area, $C_A$, and confirm that the effect of the dipole-field interaction becomes negligible when $\mu E << kT$.

c) The probability of finding a two-dimensional dipolar molecule in a state described by the phase space variables $(x, y, p_x, p_y, \theta)$ is proportional to $e^{\beta H}$. Show that if the electric field is not uniform, then the gas is more dense where the field is higher.

Hints: The modified Bessel function $I_0(u)$ is a monotonically increasing function of $|\mu|$, with an integral representation:

$$I_0(u) = \frac{1}{\pi} \int_0^\pi e^{u \cos \theta} d\theta$$

Its power series expansion is:

$$I_0(u) = \sum_{k=0}^\infty \frac{(u/2)^{2k}}{(k!)^2},$$

while for large $u$:

$$I_0(u) \sim e^u / \sqrt{2\pi u}.$$
A spin-1/2 particle is in a spin-up eigenstate $|+\rangle_z$ of $\hat{J}_z$, i.e., $\hat{J}_z |+\rangle_z = \frac{1}{2} |+\rangle_z$. Now consider the other two spin-component operators $\hat{J}_x$ and $\hat{J}_y$.

(a) What are the probabilities $|\alpha_x|^2$ and $|\alpha_x|^2$ that a measurement of $\hat{J}_x$ on $|+\rangle_z$ yields each of the respective eigenvalues of $\hat{J}_x$?

(b) What is the average value $\langle + |\hat{J}_x| + \rangle_z$ of $\hat{J}_x$ measurements on $|+\rangle_z$?

(c) What is the average value $\langle + |\hat{J}_x^2| + \rangle_z$ of $\hat{J}_x^2$ measurements on $|+\rangle_z$?

(d) Answer the same questions (a), (b), (c) for the measurements on $|+\rangle_z$ of the $\hat{J}_y$ and $\hat{J}_y^2$ operators.

(e) Now consider $\hat{J}^2 = \hat{J}_z^2 + \hat{J}_x^2 + \hat{J}_y^2$. What is the average value $\langle + |\hat{J}^2| + \rangle_z$ of $\hat{J}^2$ measurements on $|+\rangle_z$?

(f) If the spin is a classical vector oriented along the z-direction, what are the differences in the answers to questions (a) - (e)? What is the physical origin of these differences?

(g) Now consider a spin-3/2 particle in a spin-3/2 eigenstate of $\hat{J}_z$, i.e., $\hat{J}_z |\frac{3}{2}\rangle_z = \frac{3}{2} \frac{3}{2} |\frac{3}{2}\rangle_z$. Consider the probability amplitudes $\alpha_z = \langle x |\frac{3}{2}\rangle_z$ and $\beta_z = \langle x |\frac{3}{2}\rangle_z$ that $\hat{J}_x$ has eigenvalues $\left( \pm \frac{3}{2} \right)$ and $\left( \pm \frac{1}{2} \right)$ on $|\frac{3}{2}\rangle_z$, respectively, and likewise for $\hat{J}_y$. Relying upon symmetry consideration, and the same logic as in the previous analysis of the spin-1/2 case, determine the probabilities $|\alpha_z|^2$ and $|\beta_z|^2$. 

PhD/MS Comprehensive Examination
Spring 2006
Quantum Mechanics 600-1
The carbon dioxide molecule has a linear O=\text{C}=\text{O} form in equilibrium. Let \( a \) be the C=O equilibrium distance and \( f \) the force constant of the C=O vibration. Allow for different oxygen isotopes coupling to the carbon atom, hence for different masses \( m_1, m_2, m_3 \) (note, however, that \( a \) and \( f \) are independent of oxygen isotope).

Determine the two valence vibration frequencies in the harmonic approximation using a one-dimensional model, thus ignoring bending vibrations.

You may use the Schrödinger picture (wave mechanics) or the Heisenberg picture (operator methods) to solve this problem.

a) Set up the Schrödinger equation or Hamiltonian for the linear harmonic oscillator model.

b) Separate the center-of-mass motion.

c) Remove any cross terms in the kinetic (and potential) energy, thus allowing for factorizing the internal motion into two harmonic oscillators.

\textit{Hint:} Rotation in ket space (similar to principal axes transformation).

d) Determine their eigenvalues and ground-state wavefunction for the cases of different and same oxygen isotopes.
A non-relativistic spinless, particle of mass \( m \) and charge \( q \) is constrained to move in a circle of radius \( R \), which lies in the \( x-y \) plane.

a) Find the normalized stationary states and corresponding allowed energies for this system. Identify all degeneracies.

A very strong electric field \( \vec{E} \) (\( qER \gg \hbar/(mR^2) \)) is now introduced in the plane of the circle along the \( x \)-direction.

b) Write out the Hamiltonian for this system

c) By making an appropriate "small-angle approximation," and explaining why this is reasonable, obtain an approximate Hamiltonian for which the Schrodinger equation can be solved exactly. Find the eigenvalues for this Hamiltonian.

d) Improve your result in part (c) by retaining the next-higher term in the small-angle approximation and treating this as a perturbation. Use first-order perturbation theory to obtain the correction to the ground-state energy.