CUA

THE CATHOLIC UNIVERSITY OF AMERICA

Department of Physics
200 Hannan Hall
Washington, D.C. 20064
202-319-5315
Fax 202-319-4448

RULES FOR THE PHYSICS COMPREHENSIVE EXAMINATION

SPRING 2005

In order to assure an equitable and fair Comprehensive Examination for all students the following rules will apply to the examination administered by the Department of Physics.

1. The examination will be closed book. Students will not be permitted to bring to the examination room materials such as notes, reference books, calculators, or other aids of any form. The Physics Department will supply copies of the *CRC Mathematical Handbook,* Schaum's *Mathematical Handbook, Table of Functions* by Jahnke and Emde, and the NBS *Handbook of Mathematical Functions*, for the use of those taking the exam should they feel the need for these references during the examination.

   Students who want to use their own mathematical tables and/or handbooks must get permission from the Chairman prior to the beginning of the comprehensive examination. These tables and/or handbooks will remain under the control of the department until after the examination is completed, but will be available to their owners during the exam.

2. If needed, students will be allowed a short break, not to exceed ten minutes, during the examination period. While absent from the examination room students may not consult material or other sources of information concerning any matter pertaining to the examination. Students who leave the examination room will turn in all of their examination materials to the proctor before leaving the room. Examination papers will be returned to the students when they reenter the examination room. Students who fail to return within ten minutes will be considered to have completed that portion of the examination.

   Only one student will be permitted to leave the examination room at a time.

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Fax 202-319-4448

PhD/MS/PRELIM. Comprehensive Examination

Spring 2005

Thursday, March 17, and Friday, March 18, 2005

THE WRITTEN EXAM WILL BE HELD IN ROOM

133 HANNAN HALL

THE EXAM WILL BE FROM 9:00 AM - 5:00 PM ON THURSDAY March 17, AND 9:00 AM - 5:00 PM ON FRIDAY, March 18, 2005.
THERE WILL BE A GET TOGTHER ON THE 2^{ND} FLOOR AT 5:00 PM ON FRIDAY.

For the PhD students:
THE ORAL PORTION OF THE PhD COMPREHENSIVE EXAM WILL BEGIN THE FOLLOWING WEEK.

PLEASE CALL (202) 319-5315) OR STOP BY THE PHYSICS DEPARTMENT OFFICE ON FRIDAY TO INQUIRE ABOUT YOUR SCHEDULED ORAL EXAM.

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Department of Physics
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Washington, D.C. 20064
202-319-5315
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Preliminary Examination

Spring 2005

Thursday, March 17, and Friday, March 18, 2005

Room 133 - Hannan Hall

- **YOU MUST DO TWO QUESTIONS IN EACH OF THE AREAS:**

Thursday, March 17, 2005

Mechanics

Electricity & Magnetism

Friday, March 18, 2005

Thermodynamics

Modern Physics/Quantum Mechanics

- **DO EACH PROBLEM IN A SEPARATE BLUE BOOK**

- **PUT YOUR NAME ON EACH BLUE BOOK**

- **LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS**
  FOR EXAMPLE: Mechanics #1

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200 Hannan Hall
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202-319-5315
Fax 202-319-4448

MS Comprehensive Examination
Physics Department

Spring 2005

Thursday, March 17, and Friday March 18, 2005

Room 133 - Hannan Hall

GENERAL INSTRUCTIONS:

This examination is divided into four sections as follows:

Thursday, March 17, 2005	
9:00 a.m. - 12:00 Noon	Classical Mechanics - 2 questions
1:00 P.M. - 5:00 P.M.	E & M - 2 questions
Friday, March 18, 2005	
9:00 a.m. - 12:00 Noon	Thermodynamics/Stat. Mech. - 2 questions
1:00 P.M. - 5:00 P.M.	Modern Physics/Quantum Mech. - 2 questions

DO EACH PROBLEM IN A SEPARATE BLUE BOOK

PUT YOUR NAME ON EACH BOOK

LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS
FOR EXAMPLE: Mechanics 600-1 or Electromagnetism 500-2

YOU MUST DO TWO (AND ONLY TWO) OF THE QUESTIONS IN EACH AREA.

OF THE TOTAL OF EIGHT QUESTIONS THAT ARE DONE, A MINIMUM OF TWO MUST BE FROM THE 600-LEVEL QUESTIONS.

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PhD Comprehensive Examination

Physics Department

Spring 2005

Thursday, March 17, and March 18, 2005

Room 133 - Hannan Hall

GENERAL INSTRUCTIONS:

This examination is divided into four sections as follows:

Thursday, March 17, 2005

9:00 a.m. - 12:00 Noon	Classical Mechanics - 2 questions
1:00 P.M. - 5:00 P.M.	E & M - 3 questions

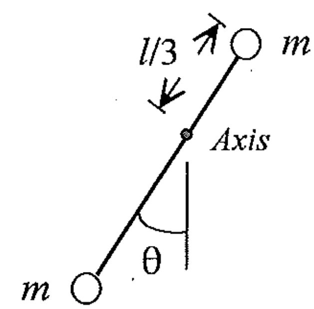
Friday, March 18, 2005

9:00 a.m. - 12:00 Noon	Stat Mech. - 2 questions
1:00 P.M. - 5:00 P.M.	Quantum Mech. - 3 questions

DO EACH PROBLEM IN A SEPARATE BLUE BOOK

PUT YOUR NAME ON EACH BOOK

LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS
FOR EXAMPLE: Mechanics 600-1



Two equal masses m are connected by a massless rigid rod of length l, as shown. The system is free to rotate about a horizontal axis perpendicular to the rod, which passes through the rod at a distance $l/3$ from the closer mass.

a) Find the distance of the center of mass from the axis of rotation.

b) Calculate the moment of inertia of the system about the axis of rotation.

c) Write the potential energy of the system as a function of the angle θ.

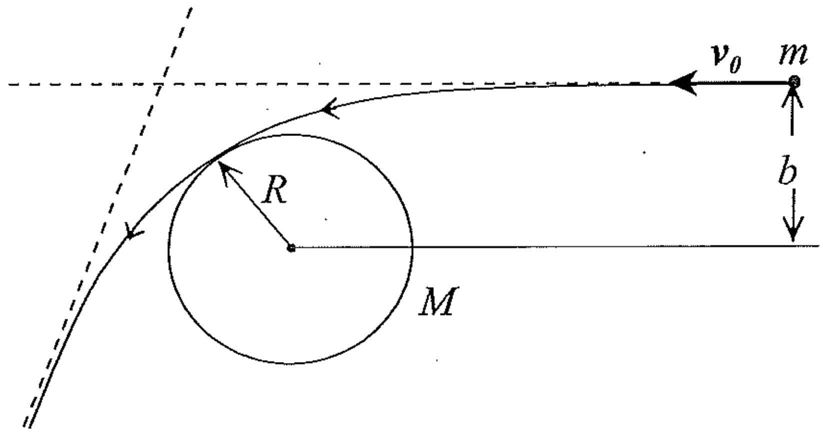
d) Set up the Lagrange equation(s) of motion for the system, using the angle θ as the generalized coordinate.

e) Considering only small angles θ, find the frequency of oscillation about equilibrium.

f) Starting at the equilibrium position, what *minimum angular velocity* must be given to the system if it is to continue in rotation instead of oscillating?

MS Comprehensive/Preliminary Examination
Spring 2005
Classical Mechanics: CM500-2

An unpowered spacecraft of mass m is approaching a planet of mass $M \gg m$ and radius R. When the spacecraft is very far away from the planet, its velocity is $\vec{v}_0$ and its impact parameter is b (see the figure).



a) What is the angular momentum of the spacecraft about the center of the planet?

b) Find b_{min}, the minimum value of b for which the spacecraft will just miss hitting the planet. Note that for this condition, the perihelion of the spacecraft's orbit is just outside the surface of the planet.
Hint: Use energy conservation and the effective potential appropriate for the spacecraft's orbit. Assume that the center of the planet is at rest.

c) Discuss the limits of b_{min} in the limiting cases $v_0^2 \gg \frac{GM}{R}$ and $v_0^2 \ll \frac{GM}{R}$.

MS Comprehensive/Preliminary Examination
Spring 2005
Electricity and Magnetism: EM 500-1

A non-conducting sphere of radius R contains a non-uniform charge density

$$\rho(r) = Ar \quad (0 \leq r \leq R).$$

a) Find the total charge of the sphere.

b) Find the electric field at any point *outside* the sphere ($r > R$).

c) Find the electric field at any point *inside* the sphere ($0 < r < R$).

d) Find the **potential** of the surface of the sphere ($r = R$), taking $V = 0$ at infinity.

e) Find the **difference of potential** between the center of the sphere and the surface of the sphere.

A magnet creates a magnetic field with y and z components given by $B_y = kx$, and $B_z = 0$, where k is a constant.

a) Calculate $B_x(x, y, z)$ if $\mathbf{B} = 0$ at position (0,0,0).

b) A proton with an initial velocity v_z (only in the z-direction) enters the magnetic field region at position $(x_0, y_0, 0)$. Calculate the initial force on the proton.

c) Find the equations for the trajectory of the proton, valid for as long as the transverse momentum remains negligible compared to the initial momentum.

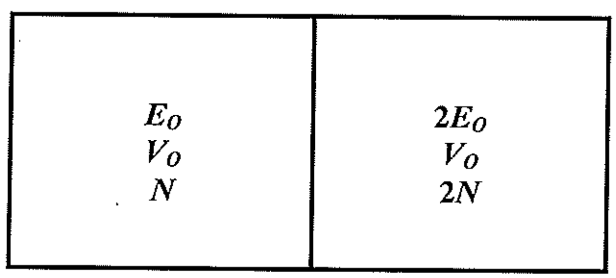
Consider a gas, which you cannot assume to be ideal, that obeys the fundamental relation in the entropy representation:

$$S = aV^{1/2}(NE)^{1/4}$$

where V is the volume, N is the number of moles, E is the energy, and a is a constant. An isolated cylinder is separated into two equal halves, each of volume V_0, by a fixed partition, as shown in the diagram. N moles of the gas with energy $E_L=E_0$ are placed in the left half and $2N$ moles of the gas with energy $E_R=2E_0$ are placed in the right half.

(a) Assuming that the partition conducts only heat, (i) State the condition for equilibrium and (ii) Find the energy in the left and right compartments at equilibrium in terms of E_0.

(b) Assuming that the partition conducts heat and also moves freely, (i) State the conditions for equilibrium and (ii) Find the volumes and energies of the samples in each partition at equilibrium in terms of V_0 and E_0.



(a) Give a simple physical reason why the heat capacity at constant pressure (C_p) is greater than the heat capacity at constant volume (C_V).

(b) Show that

$$C_p - C_V = VT\frac{\alpha^2}{\kappa},$$

where V is the volume, T is the absolute temperature, α is the thermal expansion coefficient and κ is the isothermal compressibility.

(c) Calculate α and κ, and hence C_p - C_V, for an ideal gas.

MS Comprehensive exam: If you choose this problem, you may not also choose QM 600-2

MS Comprehensive/Preliminary Examination
Spring 2005
Quantum Mechanics/Modern Physics: QM 500-1

The spin state of an electron can be described by a "spinor" wave function $\chi = a\chi_+ + b\chi_-$

where a and b are complex numbers, and χ_+ and χ_- are orthonormal basis vectors in a two-dimensional space. χ_+ and χ_- obey the following eigenvalue equations:

$$S_z\chi_+ = \frac{\hbar}{2}\chi_+, \quad S_z\chi_- = -\frac{\hbar}{2}\chi_-,$$

where S_z is the operator representing the z-component of spin angular momentum vector

$$\boldsymbol{S} = \hat{\imath}S_x + \hat{\jmath}S_y + \hat{k}S_z.$$

The spinors can be represented by column matrices:

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and the operators by two-by-two matrices. For example, the operators corresponding to the x and y components of the spin angular momentum have the matrix representation

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

a) Write the matrices representing the operators S_z and $S^2 = S_x^2 + S_y^2 + S_z^2$ in this representation.

b) Write the matrices $S_x + iS_y$ and $S_x - iS_y$, and show that they act as raising and lowering operators when applied to the basis spinors χ_+ and χ_-.

Note: Part (c) and (d) do not depend on the answers to part (a) or (b).

c) Consider an electron with orbital angular momentum quantum number $\ell = 2$. If you measured the following quantities, what values might you obtain?

i) L^2, the square of the orbital angular momentum
ii) L_z, the z component of the orbital angular momentum

d) Consider **two electrons** in an atom which each have orbital angular momentum quantum number $\ell = 2$ and spin quantum number $s = ½$. What are the possible values for the following quantum numbers?

i) ℓ_{tot}, the total orbital angular momentum of both electrons.
ii) s_{tot}, the total spin angular momentum of both electrons.
iii) j_1, the total angular momentum (spin and orbital) of electron 1.
iv) j_{tot}, the total angular momentum (spin and orbital) of both electrons.

A particle of mass m is confined to one-dimensional motion between two walls at $x = 0$ and $x = L$, whose corresponding infinite-well potential is:

$$V = 0, \text{ for } 0 \leq x \leq L$$
$$V = \infty, \text{ for } x < 0, x > L$$

a) Determine the first and second lowest energy eigenvalues, E_1 and E_2 and the corresponding normalized eigenfunctions u_1 and u_2 of the Hamiltonian.

b) At initial time $t = 0$, the wavefunction is given as $\Psi(x,0) = \frac{1}{\sqrt{2}}[\, u_1(x) + u_2(x)]$. Determine the time-evolution of the wavefunction, $\Psi(x,t)$, at later times t.

c) For the wavefunction $\Psi(x,t)$ determined in (b), determine the probability $P(t)$ that the particle can be found in the region $0 \leq x \leq \frac{L}{2}$ as a function of time.

Consider the motion of a point mass m in the gravitational field of a much larger mass M. General relativity theory predicts a correction to Newtonian theory, such that the effective potential $V_{eff}(r)$ which enters the radial equation for conservation of energy,

$$\frac{1}{2} m \left(\frac{dr}{dt} \right)^2 + V_{eff}(r) = \mathscr{E}^* = \text{const},$$

becomes $$V_{eff}(r) = -\frac{GmM}{r} + \frac{L^2}{2mr^2} - \frac{GML^2}{mc^2 r^3}.$$

If you prefer, you may use units in which $G = 1$ and $c = 1$.

a) Sketch $V_{eff}(r)$ as a function of r. In this sketch (only) you may ignore the "post-Newtonian" term ($-GML^2/mc^2r^3$), regarding it as a small perturbation.

b) Show that for a **circular orbit** of radius r_0, we must have $\left. \frac{dV_{eff}}{dr} \right|_{r_0} = 0$, and determine the corresponding L_0^2 in terms of r_0 and M. What is the effect of the post-Newtonian term on L_0^2?

c) Consider a **small perturbation of a stable circular orbit, preserving** L_0. Show that r performs small oscillations around r_0 with a radial angular frequency

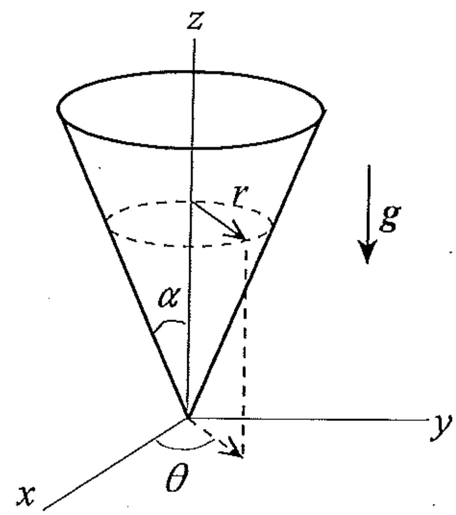
$$\omega_r = \left[\frac{1}{m} \left. \frac{d^2 V_{eff}}{dr^2} \right|_{r_0} \right]^{1/2}.$$

Determine ω_r^2 in terms of r_0 and M. What is the effect of the post-Newtonian term on ω_r^2? Conclude that no stable circular orbit is possible for $r_o \le 6GM/c^2$.

d) Now determine the **angular frequency of rotation** of the unperturbed circular orbit, i.e. $\omega_0 = \frac{d\varphi}{dt} = \frac{L_0}{mr_0^2} = \text{const}$, in terms of r_0 and M. How does this differ from Kepler's 3rd law?

e) Argue that the **perihelion** of the perturbed orbit must advance at a precession rate $\Delta\omega = \omega_0 - \omega_r = \omega_r \left(\frac{\omega_0}{\omega_r} - 1 \right)$, or, equivalently, that the point mass sweeps out an angle $\varphi = \omega_0 \tau_r = 2\pi \frac{\omega_0}{\omega_r} = 2\pi \left(\frac{\omega_0}{\omega_r} - 1 \right) + 2\pi \equiv (\Delta\varphi)_{shift} + 2\pi$ in each cycle of small radial oscillations. Determine the corresponding angular shift of the perihelion per cycle, $(\Delta\varphi)_{shift}$.

A particle of mass m is constrained to move on a frictionless conical surface of half-angle $\alpha < \frac{\pi}{2}$. The axis of the cone is in the $+z$ direction, and there is a uniform gravitational field in the $-z$ direction. Choose (r, θ) as the Lagrangian coordinates (see figure), and use the constraint equation $z = r \cot \alpha$.



a) Determine the Lagrangian and the Lagrange equations of motion for the particle.

b) Find two constants of the motion, and interpret their physical meaning.

c) Based on those findings, find the formal solution $t = t(r)$ by quadrature of the equivalent one-dimensional problem.

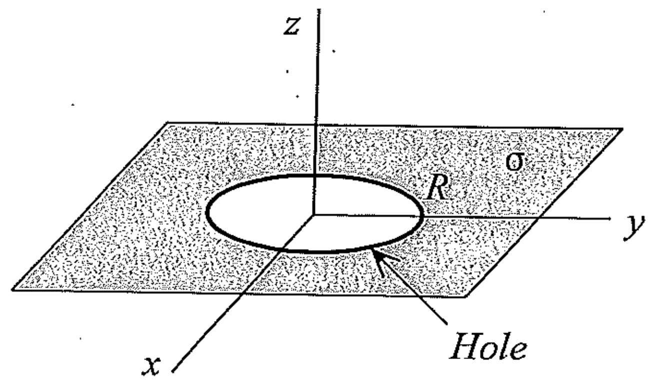
d) Show that a circular orbit at $r = r_0 = \text{const}$ is possible, and determine the corresponding angular velocity $\dot{\theta}$.

e) Consider a small radial perturbation

$$\delta r = r(t) - r_0 = (\delta r)_0 \cos(\omega_r t)$$

from the circular orbit at $r = r_0 = \text{const}$. Determine whether the perturbed motion is stable, and, if so, find its radial angular frequency of oscillation ω_r in terms of r_0, g and α.

An infinite insulating plane of negligible thickness has a **circular hole** of radius R, and has an otherwise uniform surface charge density σ. The z axis is normal to the plane, and passes through the center of the hole.



a) Find the electrostatic potential $\Phi(0, 0, z)$ along the z axis. (Hint: Since the total charge is infinite, you must choose an arbitrary reference point – NOT infinity – for $\Phi=0$.)

b) Find the potential $\Phi(r<R, \theta)$ anywhere inside the sphere of radius R bounded by the circular hole. Retain only the first three non-vanishing terms in the expansion of $\Phi(r<R,\theta)$. (Hint: $\sqrt{1+x} = 1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\ldots$)

Consider a homogeneous linear medium with constant permittivity ϵ, permeability μ and conductivity σ:

$$\boldsymbol{B} = \mu \boldsymbol{H}, \quad \boldsymbol{D} = \epsilon \boldsymbol{E}, \quad \boldsymbol{J}_{free} = \sigma \boldsymbol{E}.$$

Assume that the medium remains electrically neutral ($\rho_{free} = 0$), and that μ, ϵ, and σ are real constants.

a) Starting with Maxwell's equations, show that the electric field obeys the wave equation

$$\nabla^2 \boldsymbol{E} - \epsilon\mu \frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \sigma\mu \frac{\partial \boldsymbol{E}}{\partial t} = 0,$$

and that the magnetic field obeys the same equation.
Hint: Use the vector identity $\nabla\times(\nabla\times \boldsymbol{F}) = \nabla(\nabla\cdot \boldsymbol{F}) - \nabla^2 \boldsymbol{F}$.

Consider a plane monochromatic wave in the z direction,

$$\boldsymbol{E}(z,t) = \hat{z}\, E_0\, e^{i(kz-\omega t)}$$

where E_0 is a complex constant. (The physical field is the real part of $\boldsymbol{E}$.)

b) Derive an expression for the wave number k in terms of μ, ϵ, σ and ω. Show that k must be complex when $\sigma\neq 0$.

c) Let the wave number be written $k = k_r + i k_i$ (where k_r and k_i are real numbers). Determine $k_r^2 - k_i^2$ and $k_r k_i$. You do not need to solve for k_r and k_i in general.

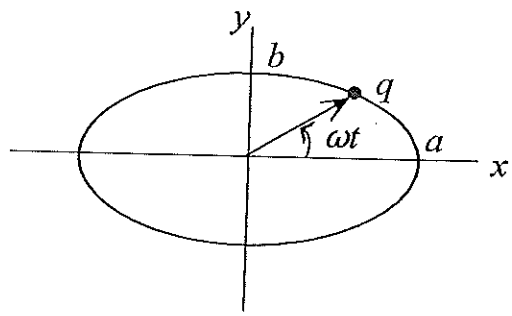
d) For a "good conductor" (when σ is sufficiently large – how large?), show that $k_i \approx k_r$, and find k_i and k_r in this approximation.

e) Show that the expression for the electric field can be written as the product of a sinusoidal wave and an attenuation factor. In the approximation of part (d), find the "skin depth" (characteristic penetration distance) for the wave in this medium. Does the skin depth increase or decrease with frequency?

f) In the approximation of part (d), find the **ratio of amplitudes** $\frac{|B_{max}|}{|E_{max}|}$ and the **phase difference** $\delta_B - \delta_E$ between the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ of the wave. (Hint: consider $\nabla\times\boldsymbol{E}$ for the plane wave given above.) What are the corresponding values for waves in free space?

A point charge q moves along an ellipse of semi-major and semi-minor axes (a, b) in the x-y plane with constant angular velocity $\omega \ll c/a$, *i.e.* tracing out the trajectory

$$x(t) = a\cos(\omega t), \quad y(t) = b\sin(\omega t) .$$



a) Determine the dipole moment $\vec{p}(t) = \vec{p}\, e^{-i\omega t}$ of the orbiting charge. As usual, it is assumed that the real parts of such expressions must be taken to obtain the corresponding physical quantities. Hence, show that $\vec{p} = q(a\hat{e}_x + ib\hat{e}_y)$, where $\hat{e}_x$ and $\hat{e}_y$ are unit vectors in the x and y directions.

(b) Knowing that $\vec{A}(\vec{r},t) \simeq -i\kappa\vec{p}\,\frac{e^{i\kappa r}}{r}\,e^{-i\omega t}$, determine the fields $\vec{B}(\vec{r})e^{-i\omega t}$ and $\vec{E}(\vec{r})e^{-i\omega t}$ in the far zone ($r \gg \frac{1}{\kappa} = \frac{c}{\omega}$) and in the dipole approximation.

(c) Determine the polarization $(\hat{n}\times\vec{p})\times\hat{n}$ of the radiated wave. Discuss the polarization in particular along the x, y and z axes.

(d) Determine the time-averaged radiated power per solid angle, $\frac{dP}{d\Omega}$. Discuss in particular the limits of (i) $a=0$, (ii) $b=0$, and (iii) $a=b=R$, in terms of the polar angles θ_x, θ_y and θ_z, which are the angles between $\hat{n}$ and $\hat{e}_x$, $\hat{e}_y$, and $\hat{e}_z$ respectively. Show that $\frac{dP}{d\Omega}$ can be expressed as the incoherent sum of $\frac{dP_x}{d\Omega} = \frac{c}{8\pi}\kappa^4 q^2 a^2 \sin^2\theta_x$ and $\frac{dP_y}{d\Omega} = \frac{c}{8\pi}\kappa^4 q^2 b^2 \sin^2\theta_y$.

(e) Determine the total time-averaged radiated power $P = \int d\Omega\, \frac{dP}{d\Omega}$.

Black-body radiation may be viewed as an ideal gas of photons, with a number of photons $N_e = \int_0^\infty \frac{g(\epsilon)\,d\epsilon}{e^{\beta\epsilon} - 1}$ in the excited states (or "gas phase"), and an undetermined number of photons in the ground state with $\epsilon = 0$ (the "condensed phase"). Equivalently, the two "phases" are in equilibrium at $z = e^{\mu/kT} = 1$.

The density of states $g(\epsilon)$ reflects the energy-momentum relation $\epsilon = c\,|\vec{p}|$ for zero-mass photons, as well as their $g_S = 2$ "spin degeneracy."

a) Determine $g(\epsilon)$ and N_e, in terms of V, T, and the constants h, c and k.

The pressure and energy of the photon gas can be obtained from the grand-canonical relations

$$\frac{PV}{kT} = \log Z = -\int_0^\infty g(\epsilon)\log\left(1 - e^{-\beta\epsilon}\right)d\epsilon$$

and

$$U = -\frac{\partial}{\partial\beta}(\log Z)_V = \int_0^\infty \frac{\epsilon\, g(\epsilon)\,d\epsilon}{e^{\beta\epsilon} - 1}.$$

b) Determine P and U, and the relation between them, in terms of V, T, and constants.

The equilibrium at $z = 1$ between the "gas phase" and the "condensed phase" is expected to obey the Clapeyron equation, $\frac{dP}{dT} = \frac{T\Delta s}{T\Delta v}$.

c) Interpret and determine Δs and Δv, and verify that the Clapeyron equation is obeyed by the photon gas/condensate system.

Useful formulae: $\int_0^\infty \frac{x^{n-1}\,dx}{e^x - 1} = \Gamma(n)\zeta(n)$, $\zeta(3) \simeq 1.202$, $\zeta(4) \simeq 1.082$.

Consider a system of N hard-sphere particles of diameter b that are constrained to move in one-dimension (i.e., along a line). Let the coordinate of the left side of the i-th particle be x_i. The length of the system is fixed such that $0 \leq x_i \leq L$ for all i.

(a) Obtain the partition function $Q_N(\beta, L)$ and from it, the Helmholtz free energy, the internal energy, and the heat capacity at constant length.
[Note: Since the particles cannot pass through one another, they must be treated as if they are distinguishable.]

(b) Show that the equation of state for the pressure is the Tonks equation of state (Tonks, 1936):

$$p = \frac{NkT}{x_N - (N-1)b}.$$

(c) Show that if a constant, infinite-ranged two-body attractive interaction $u_{ij} = -2a(N-1)/L$ is added, the equation of state for the system becomes the one-dimensional van der Waals equation, for $N >> 1$.

Now consider the original system (*not* the one in part (c)) but with the constraint of constant length replaced by a constraint of constant force F. Namely, the rightmost particle is subjected to a constant force F towards the origin. The force is derivable from the potential $U=F(x_N - (N-1)b)$ such that $U=0$ corresponds to the lowest energy state of the system.

(d) Write the Hamiltonian for the system.

(e) Calculate the partition function $Y_N(\beta, F)$.

(f) Calculate the enthalpy and the heat capacity at constant force.

(g) Calculate the equilibrium position of the rightmost particle (the equilibrium length of the system). Show that this leads to the same equation of state as part (b).
[Note that for a one-dimensional system, the pressure equals the force F; there is no "area" to divide by.]

Consider, in the "Schrödinger picture," time-independent operators $\hat{O}$ such as the Hamiltonian $\hat{H}$, the momentum $\hat{P}$ and the position $\hat{X}$.
In the Schrödinger picture, the states $|\Psi(t)\rangle_S$ of the system evolve in time according to

$$|\Psi(t)\rangle = e^{-i\hat{H}t}|\Psi(0)\rangle.$$

a) Verify that this time evolution satisfies the Schrödinger equation,

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$$ (where units with $\hbar = 1$ are assumed.)

In the "Heisenberg picture," the states are always brought back to the initial time, so they are time-independent:

$$|\Psi\rangle_H = e^{i\hat{H}t}|\Psi(t)\rangle = |\Psi(0)\rangle.$$

Correspondingly, the operators in the Heisenberg picture become time-dependent:

$$\hat{O}(t)_H = e^{i\hat{H}t}\,\hat{O}\,e^{-i\hat{H}t}.$$

b) Show that the operators satisfy the Heisenberg equation of motion:

$$i\frac{d}{dt}\hat{O}(t)_H = \left[\hat{O}(t)_H, \hat{H}\right].$$

c) Now consider the free-particle Hamiltonian in one dimension: $\hat{H} = \frac{\hat{P}^2}{2m}$. Find the position operator $\hat{X}(t)_H$ in the Heisenberg picture, in terms of $\hat{X}$ and $\hat{P}$, by solving the Heisenberg equation of motion, with the appropriate initial condition at $t = 0$.
Hint: Recall $\left[\hat{X}, \hat{P}\right] = i$.

(d) Show that the average values of operators agree in the two pictures, namely

$${}_H\langle\Psi\,|\,\hat{O}(t)_H\,|\,\Psi\rangle_H = \langle\Psi(t)\,|\,\hat{O}\,|\,\Psi(t)\rangle.$$

(e) Given three operators $\hat{A}$, $\hat{B}$, $\hat{C}$ in the Schrödinger picture, satisfying the commutation relation $\left[\hat{A}, \hat{B}\right] = \hat{C}$, what is the corresponding commutation relation in the Heisenberg picture?

Consider a spin-3/2 ($s = 3/2$) particle. The normalized eigenfunctions $|s\,m\rangle$ of the operators S^2 and S_z can be represented by four-dimensional column matrices

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \text{ and } \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

a). Construct matrices representing the operators S^2, S_z, S_+, S_-, S_x, and S_y in this representation.

Hint: The operators $S_\pm$ are defined by the equations

$$S_\pm = S_x \pm iS_y$$
$$S_+|s\,m\rangle = \hbar\sqrt{s(s+1)-m(m+1)}\,|s\,m+1\rangle$$
$$S_-|s\,m\rangle = \hbar\sqrt{s(s+1)-m(m-1)}\,|s\,m-1\rangle$$

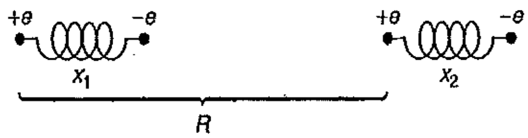
b) Of the six matrices constructed in part (a), which are hermitian?

c) An arbitrary spin state in this space can be represented by the column matrix $\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$. Find the expectation value of S_z in this state.

Part (d) does not depend on parts (a)-(c).

d) Consider a particle with spin $s = 3/2$ and orbital angular momentum $l = 2$. What are the possible quantum numbers j, m_j which describe the total angular momentum $\vec{J} = \vec{L} + \vec{S}$?

Consider the following simplified model for the van der Waals interaction between two neutral atoms. Each atom is modeled by an electron (charge $-e$, mass m) bound to a motionless nucleus of charge $+e$ by a harmonic oscillator potential with spring constant k. The nuclei are a distance R apart and the *orientation of the oscillators is fixed,* as shown in the diagram.



The Hamiltonian of the unperturbed system is then:

$$H^0 = \frac{p_1^2}{2m} + \frac{1}{2}kx_1^2 + \frac{p_2^2}{2m} + \frac{1}{2}kx_2^2.$$

The Coulomb interaction between the atoms is:

$$H' = \frac{1}{4\pi\varepsilon_0}\left(\frac{e^2}{R} - \frac{e^2}{R-x_1} - \frac{e^2}{R+x_2} + \frac{e^2}{R-x_1+x_2}\right).$$

(a) Explain the above equation for H'. Assuming that both $|x_1|$ and $|x_2|$ are much less than R, show that:

$$H' \cong -\frac{e^2 x_1 x_2}{2\pi\varepsilon_0 R^3}.$$

(b) Show that the Hamiltonian $H=H^0+H'$ separates into two Harmonic oscillator Hamiltonians under the change of variables:

$$x_\pm = \frac{1}{\sqrt{2}}(x_1 \pm x_2) \text{ and } p_\pm = \frac{1}{\sqrt{2}}(p_1 \pm p_2).$$

(c) Hence obtain the ground state energy E and compare it to the ground state energy E_0 in the absence of the Coulomb interaction. Assuming that $k >> (e^2/4\pi\varepsilon_0 R^3)$, show that:

$$\Delta V = E - E_0 \cong -\frac{\hbar}{8m^2\omega_0^3}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \frac{1}{R^6}.$$

(d) Now perform the same calculation using perturbation theory. Show that to first order $\Delta V = 0$ and that to second order you recover the same result as in part (c). You may find the following result for the harmonic oscillator useful:

$$\langle n|x|n'\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\delta_{n',n-1} + \sqrt{n'}\,\delta_{n,n'-1}\right).$$