CUA

THE CATHOLIC UNIVERSITY OF AMERICA

Department of Physics
200 Hannon Hall
Washington, D.C. 20064
202-319-5315
Fax 202-319-4448

RULES FOR THE PHYSICS COMPREHENSIVE EXAMINATION

Spring 2004

In order to assure an equitable and fair Comprehensive Examination for all students the following rules will apply to the examination administered by the Department of Physics.

1. The examination will be closed book. Students will not be permitted to bring to the examination room materials such as notes, reference books, calculators, or other aids of any form. The Physics Department will supply copies of the *CRC Mathematical Handbook,* Schaum's *Mathematical Handbook, Table of Functions* by Jahnke and Emde, and the NBS *Handbook of Mathematical* Functions, for the examinees' use should they feel the need for these references during the examination.

   Students who want to use their own mathematical tables and/or handbooks must get permission from the Chairman at least two days before the beginning of the comprehensive. These tables and/or handbooks will remain under the control of the department until after the comp exam but will be available to their owners during the exam.

2. If needed, students will be allowed a short break, not to exceed 5 minutes, to use the restroom facilities during the examination period. While absent from the examination room students may not consult material or other sources of information concerning any matter pertaining to the examination. Students who leave the examination room will turn in all of their examination materials to the proctor before leaving the room. Examination papers will be returned to the students when they reenter the examination room. Students who fail to return within 5 minutes will be considered to have completed that portion of the examination.

<u>Only one student will be permitted to leave the examination room at a time.</u>

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Ph.D/MS/PRELIM Comp. Examination

Spring 2004

Thursday, March 18, and Friday, March 19, 2004

THE WRITTEN EXAM WILL BE HELD IN ROOM 136 HANNAN HALL

THE EXAM WILL BE FROM 9:00 AM - 5:00 PM ON THURSDAY March 18, AND 9:00 AM - 5:00 PM ON FRIDAY, March 19, 2004.

THE ORAL PORTION OF THE PH.D. COMPREHENSIVE EXAM WILL BEGIN THE FOLLOWING WEEK..

PLEASE CALL (202) 319-5315) OR STOP BY THE PHYSICS DEPARTMENT OFFICE ON FRIDAY TO INQUIRE ABOUT YOUR SCHEDULED ORAL EXAM.

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200 Hannon Hall
Washington, D.C. 20064
202-319-5315
Fax 202-319-4448

MS Comprehensive Examination
Physics Department

Spring 2004

Thursday, March 18, and Friday, March 19, 2004

Room 136 - Hannan Hall

GENERAL INSTRUCTIONS:

This examination is divided into four sections as follows:

Thursday, March 18, 2004	
9:00 a.m. - 12:00 Noon	Classical Mechanics - 2 questions
1:00 P.M. - 5:00 P.M.	E & M - 2 questions
Friday, March 19, 2004	
9:00 a.m. - 12:00 Noon	Thermodynamics/Stat. Mech. - 2 questions
1:00 P.M. - 5:00 P.M.	Modern Physics/Quantum Mech. - 2 questions

DO EACH PROBLEM IN A SEPARATE BLUE BOOK

PUT YOUR NAME ON EACH BOOK

LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS
FOR EXAMPLE: Mechanics 600-1 or Electromagnetism 500-2

YOU MUST DO TWO (AND ONLY TWO) OF THE QUESTIONS IN EACH AREA.

OF THE TOTAL OF EIGHT QUESTIONS THAT ARE DONE, A MINIMUM OF TWO MUST BE FROM THE 600-LEVEL QUESTIONS.

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Ph.D. Comprehensive Examination

Physics Department

Spring 2004

Thursday, March 18, and March 19, 2004

Room 136 - Hannan Hall

GENERAL INSTRUCTIONS:

This examination is divided into four sections as follows:

<u>Thursday, March 18, 2004</u>

9:00 a.m. - 12:00 Noon	Classical Mechanics - 2 questions
1:00 P.M. - 5:00 P.M.	E & M - 3 questions

<u>Friday, March 19, 2004</u>

9:00 a.m. - 12:00 Noon	Stat Mech. - 2 questions
1:00 P.M. - 5:00 P.M.	Quantum Mech. - 3 questions

<u>DO EACH PROBLEM IN A SEPARATE BLUE BOOK</u>

PUT YOUR NAME ON EACH BOOK

LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS
FOR EXAMPLE: Mechanics 600-1

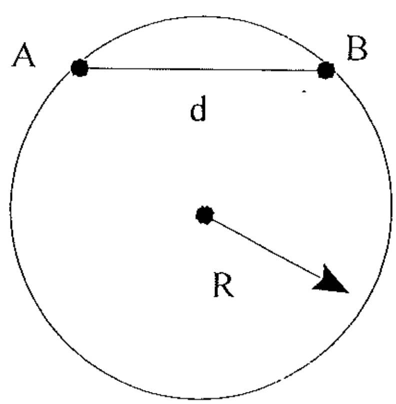
Two cities (A and B) are to be connected by a straight tunnel through the earth, as shown in the diagram. A train travels between the two cities powered only by the gravitation force of the earth (neglect friction and the motion of the earth). The length of the tunnel is d and the radius of the earth is R. Assume that the earth is a sphere of constant mass density.

a) Find the gravitational force on the train, and from it the potential energy of the train, as a function of its distance from the center of the earth, r. Assume that the train starts from rest at point A.

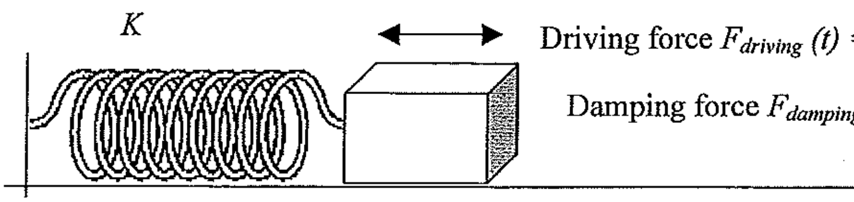
b) Using conservation of energy find the maximum speed of the train along its track through the tunnel.

c) Find the velocity of the train at any arbitrary point in the tunnel and show that the travel time from A to B is $\pi(R/g)^{1/2}$.

Hint: Recall from Newton's Theorem that a particle located within a spherical mass distribution at a distance r from the center experiences a net gravitational force equal to the contribution from the mass within the radius r only.



Consider a **damped driven harmonic oscillator** composed of a block of mass m mounted on a massless spring of spring constant K, subject to a driving force of frequency ω and a damping force proportional to the velocity of the block, as shown.



Driving force $F_{driving}(t) = F_0 \cos(\omega t)$

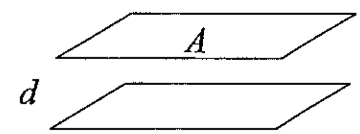
Damping force $F_{damping}(t) = -\gamma v$.

a) Write down a differential equation describing the motion of the block.

b) Solve this equation for the steady-state behavior of the system, assuming that the system oscillates at the driving frequency ω, and find both the amplitude and phase delay. Note that complex notation may be helpful.
(*Hint*: Substitute $F_{driving}(t) = F_0 e^{-i\omega t}$.)

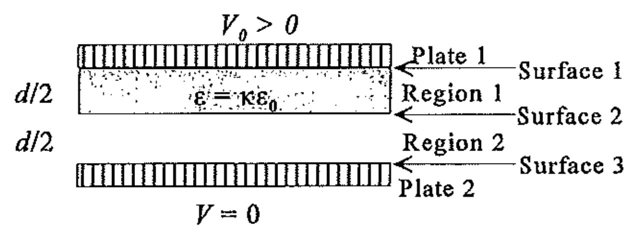
c) Describe briefly, in a few sentences, how to measure the resonance curve of the damped driven oscillator. Be specific about **what you vary** and **what you measure** and draw a rough sketch of the resulting graph.

a) Find the capacitance of an air-filled parallel-plate capacitor with plates of area A separated by a distance d. Assume (here and in all the following parts) that the plate dimensions are >> d, so that fringe fields may be neglected.



b) Recall that for a region with dielectric constant (relative permeability) $\kappa = \varepsilon/\varepsilon_0$, Gauss's law becomes $\nabla\cdot\vec{D} = \rho_{free}$, where $\vec{D} = \varepsilon_0\vec{E}+\vec{P} = \varepsilon\vec{E} = \kappa\varepsilon_0\vec{E}$.

Write the **boundary conditions** on $\vec{D}$ and $\vec{E}$ (both parallel and perpendicular components) for a boundary between linear dielectric materials of two different dielectric constants κ_1 and κ_2 , assuming that there are no "free" charges on or inside the dielectrics.



Now assume that **half** of the capacitor gap is filled (as shown) with linear dielectric material of dielectric constant κ, and the two capacitor plates are held at potential difference V_0 , with the upper plate at the higher potential.

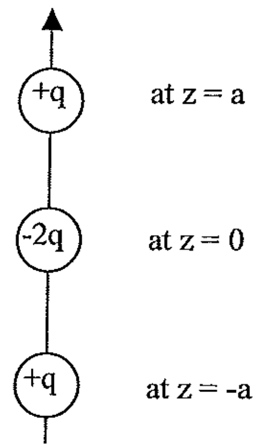
c) Find the capacitance of this capacitor (as a ratio to the capacitance C_0 of the air-filled capacitor of part (a)).

d) Find the electric field $\vec{E}$ and the $\vec{D}$ vector in Region 1 and Region 2 (in terms of V_0, d, A and κ.) Let the sign be positive for upward fields and negative for downward fields.

e) Find the **free** and **bound** surface charge density on each surface (in terms of V_0, d, A and κ.) :

Surface 1 (between dielectric and upper plate)
Surface 2 (between dielectric and air)
Surface 3 (between air and lower plate)

Consider three point charges along the z axis:



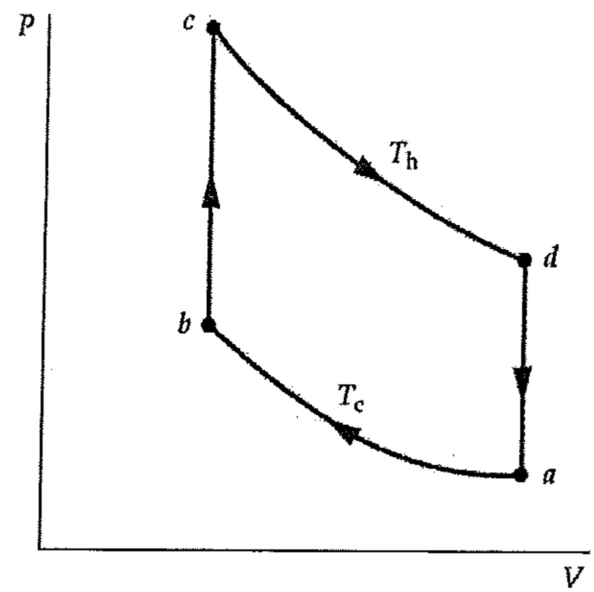
a) Calculate the potential at a point on the positive z axis for z>>a.

b) Expand the result from part (a) in powers of z to show that the lowest non-vanishing multipole moment is a quadrupole.

c) Imagine that the three point charges shown are all connected by insulating rods to make a rigid structure. If placed in a uniform static electric field, would this configuration of point charges experience a net force? A net torque? Explain your reasoning.

d) Draw a non-collinear [*i.e.* not all lying along one line] configuration of point charges that also has a quadrupole moment as its lowest non-vanishing multipole moment.

In the Stirling cycle shown in the figure process ***ab*** is an isothermal compression, process ***bc*** is heating at a constant volume, process ***cd*** is an isothermal expansion, and process ***da*** is cooling at constant volume.



a) Find the efficiency of the Stirling cycle in terms of the temperatures T_h and T_c and the volumes V_a and V_b.

b) Compare the efficiency of the Stirling cycle and the Carnot engine operating between the same maximum and minimum temperatures.

A new form of matter called compsium obeys the equation of state $p = AT^3/V$, where p is the pressure, V is the volume, T is the absolute temperature, and A is a constant. The internal energy of compsium is

$$U = BT^n \ln(V/V_o) + f(T)$$

where B, n, and V_o are all constants and $f(T)$ depends on T only. Assume that compsium can do only p-V work.

a) Write down differential expressions for the first and second laws of thermodynamics.

b) Prove the Maxwell relation $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$

c) Use these results to find B in terms of A and the value of n.

M.S. Comprehensive / Preliminary Examination
Spring 2004
Quantum Mechanics 500-1

a) Write the possible wave functions $\psi_1(x), \psi_2(x), \ldots$ for a particle of mass m confined to a one-dimensional "box" (infinite square well) of length L, extending from $x = 0$ to $x = L$:

$$V(x) = 0 \quad 0 \le x \le L$$
$$= \infty \quad x<0,\ x>L$$

Check that each function is normalized.

b) Find the energy of a particle in the ground state (described by wave function ψ_1) and in the first excited state (ψ_2).

c) Find the probability that the particle will be found in the "middle half" of the box, i.e. $\frac{L}{4} \le x \le \frac{3L}{4}$, for

i) the ground state wave function ψ_1

ii) the first excited state wave function ψ_2.

Sketch both wave functions, and verify that your calculations are sensible.

Note: Parts (d) and (e) do not depend on parts (b)–(c).

d) Write the wave function $\psi(x_1, x_2)$ for **two particles** of mass m in the box of length L, where x_1 and x_2 are the positions of the individual particles. Write your answer in terms of the single-particle functions ψ_n, assuming that one particle is in the ground state and the other particle is in the first excited state, and that the two particles are **distinguishable.**

e) Do the same as part (d) for the case that the two particles are **indistinguishable fermions** (e.g. two electrons), including the requirements of the Pauli exclusion principle.

M.S. Comprehensive / Preliminary Examination
Spring 2004
Quantum Mechanics 500-2

The following questions can be answered independently; however, answer them all for full credit.

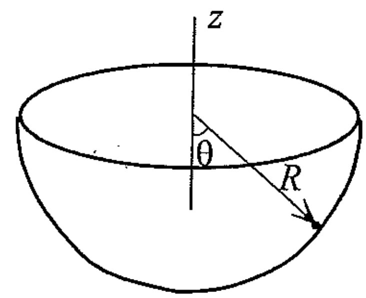
a) State the Bohr quantization postulate and from it derive the allowed energies in the Bohr model of the hydrogen atom.

b) Consider a "rigid rotor" composed of a massless rod of length r with a mass m attached to each end. The rotor is constrained to rotate in a fixed plane. Write out the Schroedinger equation for this system and obtain its allowed energies.

c) Consider the same "rigid rotor" of Part (b) but now free to assume any orientation (θ, φ).
   (i) Write out the Schroedinger equation for this system and show how it can be separated into differential equations for θ and φ separately.
   (ii) From the classical relationship between the kinetic energy and angular momentum of this system, obtain the allowed energies from the properties of the square of the angular momentum operator.

d) A sodium atom contains 11 electrons.
   (i) Assuming that the ordering of the available energies is the same as that of the hydrogen atom, construct a table showing all four quantum numbers for each of the 11 electrons in order of energy. Identify all degeneracies in this simple model.
   (ii) What effects (a) internal to the atom and (b) external to the atom might break these degeneracies?

A particle of mass m (of negligible size) slides frictionlessly on the inside of a hemispherical bowl of radius R. The motion is most easily described in terms of spherical polar coordinates r, θ, φ where r = radius of bowl is constant for the motion, θ is the polar angle relative to the vertical direction, and φ is the azimuth.



a) Write the Lagrangian in terms of θ and φ .

b) Look for any "cyclic" or "ignorable" coordinates, and identify the corresponding constants of the motion.

c) Write the Lagrange equation of motion for θ. Rewrite the Lagrangian using the constants of the motion, so that it depends only on θ and its derivatives.

d) Show that there is a possible motion for which $\theta = \theta_0$ = constant, and find the conditions for such motion.

e) Determine whether the motion in (d) is stable with respect to small perturbations in θ. If it is stable, find the frequency of small oscillations.

Note: Part (f) does not depend on the answers to parts (a)-(e).

f) Write the Lagrangian in terms of r, θ and φ . Use the method of Lagrange multipliers to find the constraint force exerted by the bowl on the particle. Interpret your result in terms of elementary forces.

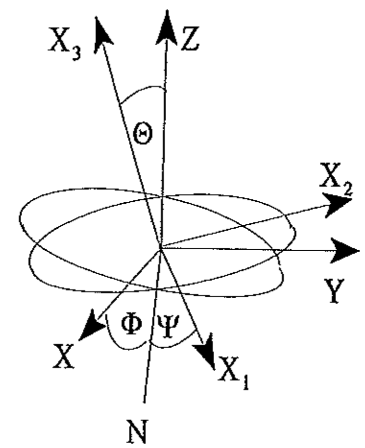
Note: Several parts of this problem can be solved also by elementary (non-Lagrangian) methods. It may be useful to check your results this way.

When a frisbee spins about an axis inclined at a small angle to its symmetry axis, it wobbles. Approximate the frisbee as a free circular disk (*i.e.* neglect gravity).

The Lagrangian for this problem is given by:

$$L(\theta,\varphi,\psi) = \frac{I_{12}}{2}(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\varphi}\cos\theta)^2$$

where I_3 is the moment of inertia with respect to the X_3-axis (normal to the disk) and $I_1 = I_2 = I_{12}$ the moment of inertia with respect to the axes in the disk's plane.



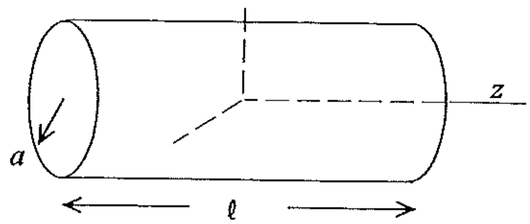
Coordinate systems: X, Y, Z are space fixed; X_1, X_2, X_3 are body fixed; N=line of nodes.

a) Solve the Euler equations for this problem.
Hint: for body fixed system: $\vec{N} = \frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L}$

b) Show that there is only precession, no nutation (i.e. θ=const).

c) Show that the rates of wobble (about space fixed Z-axis) and spin (about body fixed X_3-axis) are both constant.

d) Show that if the symmetry axis (X_3-axis) is close to the vertical (Z-axis), the rates of wobble and spin are approximately in the ratio 2:1.



a) In a material with no free currents but containing a magnetization (magnetic moment density) $\vec{M}$, the vector potential can be written

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{M}(\vec{x}')\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\, d^3x'.$$

Show that this implies that the effect of the magnetization $\vec{M}$ is to contribute an effective current density $\vec{J}_M = \vec{\nabla}\times\vec{M}$ and an effective surface current density $\vec{K}_M = \vec{M}\times\hat{n}$.

Useful mathematical identities:

$$\nabla'\frac{1}{|\vec{r}-\vec{r}'|} = \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

$$\vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A}) - \vec{A}\times(\nabla f)$$

A cylindrical bar magnet of radius a and length ℓ is permanently magnetized along its axis (the z-axis) with a uniform magnetization $\vec{M}_0 = M_0\hat{z}$.

b) Find the total magnetic dipole moment $\boldsymbol{m}$ of the bar magnet.

c) Find the effective current densities $\boldsymbol{J}_M$ and $\boldsymbol{K}_M$.

d) Consider a **solenoid** of radius a and length ℓ, with 100 uniformly-spaced turns of wire. Find the current through the solenoid which will give the same magnetic field as the bar magnet at all external points, far and near. (Hint: Consider current densities.)

e) Find the magnetic dipole moment of the solenoid of part (d). Does it agree with (b)?

f) Find the magnitude and direction of the magnetic field $\boldsymbol{B}$ at a point along the $+z$-axis at a distance 10 ℓ from the center of the magnet.
(Hint: for $r \gg \ell$ the "dipole approximation" is valid.)

Consider a hollow waveguide with perfectly conducting walls. Inside the waveguide, monochromatic waves propagate in the z direction:

$$\vec{E}(x,y,z,t) = \vec{E}_0(x,y)\, e^{i(kz-\omega t)}, \quad \vec{B}(x,y,z,t) = \vec{B}_0(x,y)\, e^{i(kz-\omega t)} \tag{1}$$

where, in general, $\vec{E}$, $\vec{E}_0$, $\vec{B}$ and $\vec{B}_0$ are *complex* vectors.
In order to fit the boundary conditions in a wave guide, the electric and magnetic fields must have *longitudinal* as well as transverse components:

$$\vec{E}_0 = E_x(x,y)\,\hat{x} + E_y(x,y)\,\hat{y} + E_z(x,y)\,\hat{z}, \quad \vec{B}_0 = B_x(x,y)\,\hat{x} + B_y(x,y)\,\hat{y} + B_z(x,y)\,\hat{z}\,. \tag{2}$$

a) Starting with Maxwell's equations, derive the wave equation satisfied by each component of $\vec{E}$ and $\vec{B}$ for an electromagnetic wave propagating in free space.
NOTE: Treat ω and k as independent variables, i.e. do not assume that $\omega/k = c$.

b) For the fields in the waveguide described by (1) and (2), derive the uncoupled partial differential equations satisfied by E_z and B_z.
Hint: The result for E_z is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2 \right) E_z = 0\,. \tag{3}$$

Note: You may begin with Equation (3) to do the following parts.

c) Consider transverse magnetic (TM) waves ($B_z = 0$, $E_z \neq 0$) of frequency ω in a perfectly conducting rectangular waveguide of height a and width b. Find the solutions for $E_z(x,y)$ by separation of variables, applying the boundary conditions $E^{\parallel} = 0$, $B^{\perp} = 0$ appropriate at the surface of a perfect conductor.

d) For given ω, find the possible wave numbers k which satisfy the above conditions. What is the lowest possible frequency ω for which a wave can propagate in a TM mode in this waveguide?

A linear antenna of length l and cross sectional area A is positioned along the z axis and centered at z=0. The antenna is fed by an alternating current $I(z,t)$ in form of a standing wave:

$$I(z,t) = I_0 \cos\frac{\pi z}{l}\sin\omega t$$

a) Using the continuity equation (with $J = \frac{I}{A}$) determine the charge density $\rho(z,t)$ and the dipole moment of the antenna.

b) For large distances ($r \gg l$) calculate the generated electromagnetic fields $\vec{E}$, $\vec{B}$ and the Poynting vector.

c) For large distances ($r \gg l$) determine the time averaged energy radiated into the solid angle dΩ.

Hints: Use spherical coordinates. For $r \gg \lambda \gg l$ the retarded potentials in Lorentz gauge are given by:

$$\vec{A} = -\frac{\mu_0 I_0 l}{2\pi^2 r}\hat{z}\sin\omega\left(t-\frac{r}{c}\right), \quad \Phi = c|\vec{A}|\cos\theta$$

Consider a system of N atoms in a magnetic field H at temperature T. Each atom has a magnetic moment μ and each magnetic moment can point either parallel or antiparallel to the field. Assume that the atoms are fixed in a lattice and that the interaction between atoms is negligible.

a) Determine the average magnetic moment $\langle M\rangle$ in the system.

b) Determine the average of $\langle\delta M\rangle^2$, where $\delta M = M - \langle M\rangle$ is the fluctuation of the total magnetic moment from its average value.

c) Derive a relation between your answer to part (b) and the susceptibility χ, where

$$\chi = \left(\frac{\partial\langle M\rangle}{\partial H}\right)_{N,T}$$

M.S. / Ph.D. Comprehensive Examination
Spring 2004
Statistical Mechanics 600-2

Consider a dilute solution, i.e. a solution with low concentration $c(n,N)=\frac{n}{N} \ll 1$, in the gravitational field near the surface of the earth.

The free enthalpy of a dilute solution is given by:

$$\Phi = N\mu_0 + n\alpha + kT\ln(n!) \approx N\mu_0 + n(\alpha + kT\ln\frac{n}{e}) = N\mu_0(P,T) + nkT\ln\frac{c(n,N)}{e} + n\psi(P,T)$$

Here $\mu_0\,(P,T)$ is the chemical potential and N the number of molecules of the pure solvent, n the number of molecules of the solute, $\alpha = f(P,T)/N$ describes the small change of enthalpy when a molecule of the solute is added to the solvent, and $\psi(P,T) = kT\ln N e^{\frac{\alpha}{kT}}$.

The chemical potentials of solvent (μ) and solute (μ') in the external (gravitational) field are given by:

$$\mu = \frac{\partial\Phi}{\partial N} + Mgz\ , \quad \mu' = \frac{\partial\Phi}{\partial n} + mgz$$

where z is the height, M the molecular mass of the solvent, and m the molecular mass of the solute.

Obtain an expression for the change of concentration as a function of height by differentiating the equilibrium conditions (T=const, μ=const, μ' =const) with respect to the height.
Assume that $\frac{dc}{dz}$ is small, *i.e.* it can be neglected, however not terms like $\frac{1}{c}\frac{dc}{dz}$, and assume that the solution is incompressible, *i.e.* you may use the molecular volumes of solvent (V) and dissolved fluid (V') in your expression: $V = \frac{\partial \mu_0}{\partial P} = const,\quad V' = \frac{\partial\psi}{\partial P} = const$

The ground-state wave function of the hydrogen-like atom with nuclear charge Z is given by

$$\psi_{100}(\vec{r}) = \frac{1}{\sqrt{4\pi}} \frac{2}{(a/Z)^{3/2}} e^{-Zr/a}.$$

The ground-state energy is $Z^2 E_{1H} = -\frac{1}{2} m c^2 \alpha^2 Z^2 = -\frac{1}{2}(e^2/4\pi\varepsilon_0) Z^2/a = -(13.6\text{ eV}) Z^2$ (where $E_{1H} = -13.6$ eV is the ground-state energy of the hydrogen atom, m = electron mass, a = Bohr radius, and α = fine-structure constant.)

a) Write the wave function for the ground state of the **helium** atom, ignoring spin, and **assuming no interaction** between the two electrons. Leave Z as a parameter in your expression, setting $Z = 2$ later as needed. What is the energy of the ground state in this approximation?

b) State the variational principle, and describe briefly how it can be used to estimate the ground-state energy of a system.

c) Use the variational principle to estimate the energy of the ground state of helium atom including the electron-electron interaction. (Ignore spin.)

As your trial wave function, take the no-interaction wave function of part (a), with the nuclear charge Z as a parameter.

Write the full Schrödinger equation, compare it with the equation satisfied by the trial function, set up the necessary integrals, and carry the analysis as far as you can.

You may use the following results:
Expectation value of the electron-electron interaction potential energy calculated using the no-interaction ground-state wave function for nuclear charge Z:

$$\langle V_{ee} \rangle = \frac{5}{4} Z \left(\frac{1}{2} m c^2 \alpha^2\right) = \frac{5}{4} Z |E_{1H}|.$$

Expectation value of $1/r$ in the ground state of a hydrogen-like atom of nuclear charge Z:

$$\left\langle \frac{1}{r} \right\rangle = \frac{Z}{a}.$$

The following part is independent of parts (a) to (c):

d) Write the **spin** part of the wave function for
(1) the ground state of the helium atom.
(2) the first excited state of the helium atom.
What are the possible orbital (l, m_l), spin (s, m_s), and total (j, m_j) quantum numbers for the total angular momentum of both electrons in each case?

A rod of length d and uniform mass distribution is pivoted at its center. The rod has mass m and a charge of $+q$ fixed at one end and a charge of $-q$ fixed at the other end. This system is to be treated quantum mechanically.

a) Write out the Hamiltonian for this system in a field-free region and find the allowed energies and eigenfunctions.

b) A constant weak electric field lying in the plane of rotation is now applied to this system. Use first order perturbation theory to find the new energies and eigenfunctions.

c) If instead the field is very strong, use first order perturbation theory to find the ground-state energy and eigenfunction.

Hint: In the strong-field case, argue that the motion will be restricted to a small range of angles.

M.S. / Ph.D. Comprehensive Examination
Spring 2004
Quantum Mechanics 600-3

Consider positronium (a bound system of electron and positron, both are spin-½ particles) in a uniform and static magnetic field along the z-axis $(\vec{B} = B_0 \hat{z})$.
The Hamiltonian describing the spin interactions is given by:

$$H = A\vec{S}^{(e+)} \cdot \vec{S}^{(e-)} + \vec{\mu}^{(e+)} \cdot \vec{B} + \vec{\mu}^{(e-)} \cdot \vec{B} \qquad with \;\; \vec{\mu}^{(e\pm)} = \frac{q}{m_e c} \vec{S}^{(e\pm)} \;, \quad A = const$$

Note: Here we ignore any effects of orbital degrees of freedom.
Assume that the spin-spin coupling (1st term) is significantly larger than the interaction between the magnetic moments and the external field.

a) Obtain the energy levels and eigenstates for the unperturbed system (i.e. B_0=0).

Hint: define $\vec{J} = \vec{S}^{(e+)} + \vec{S}^{(e-)}$ and consider $\vec{S}^{(e+)} \cdot \vec{S}^{(e-)}$ as function of $\vec{J}^2$, $\vec{S}^{(e+)2}$, $\vec{S}^{(e-)2}$.

For the following parts, $B_0 \neq 0$:

b) Obtain the energy shifts of all four possible combinations of positron and electron spin using time-independent perturbation theory (to lowest non-vanishing order).
Note: Use the same eigenket system (basis) as defined in part (a).

c) Determine the energy levels by diagonalizing the Hamiltonian matrix and compare with the solution obtained in part (b).

d) We now attempt to cause transitions (via stimulated emission and absorption) between the two m=0 states by introducing an additional magnetic field $\boldsymbol{B_1}$ oscillating at the "right" frequency.
Should we orient $\boldsymbol{B_1}$ along the z-axis or along the x- (or y-) axis?
Determine the "right" frequency of the additional field to cause these transitions.