CUA

THE CATHOLIC UNIVERSITY OF AMERICA

Department of Physics
200 Hannan Hall
Washington, D.C. 20064
202-319-5315
Fax 202-319-4448

RULES FOR THE PHYSICS COMPREHENSIVE EXAMINATION

Spring 2003

In order to assure an equitable and fair Comprehensive Examination for all students the following rules will apply to the examination administered by the Department of Physics.

1. The examination will be closed book. Students will not be permitted to bring to the examination room materials such as notes, reference books, calculators, or other aids of any form. The Physics Department will supply copies of the *CRC Mathematical Handbook,* Schaum's *Mathematical Handbook, Table of Functions* by Jahnke and Emde, and the NBS *Handbook of Mathematical* Functions, for the examinees' use should they feel the need for these references during the examination.

   Students who want to use their own mathematical tables and/or handbooks must get permission from the Chairman at least two days before the beginning of the comprehensive. These tables and/or handbooks will remain under the control of the department until after the comp exam but will be available to their owners during the exam.

2. If needed, students will be allowed a short break, not to exceed 5 minutes, to use the restroom facilities during the examination period. While absent from the examination room students may not consult material or other sources of information concerning any matter pertaining to the examination. Students who leave the examination room will turn in all of their examination materials to the proctor before leaving the room. Examination papers will be returned to the students when they reenter the examination room. Students who fail to return within 5 minutes will be considered to have completed that portion of the examination.

<u>Only one student will be permitted to leave the examination room at a time.</u>

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Department of Physics
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MS Comprehensive Examination
Physics Department

Spring 2003

Thursday, March 20, and Friday, March 21, 2003

Room 133 - Hannan Hall

GENERAL INSTRUCTIONS:

This examination is divided into four sections as follows:

Thursday, March 20, 2003

9:00 a.m. - 12:00 Noon	Classical Mechanics - 2 questions
1:00 P.M. - 5:00 P.M.	E & M - 2 questions

Friday, March 21, 2003

9:00 a.m. - 12:00 Noon	Thermodynamics/Stat. Mech. - 2 questions
1:00 P.M. - 5:00 P.M.	Modern Physics/Quantum Mech. - 2 questions

DO EACH PROBLEM IN A SEPARATE BLUE BOOK

PUT YOUR NAME ON EACH BOOK

LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS
FOR EXAMPLE: Mechanics 600-1 or Electromagnetism 500-2

YOU MUST DO TWO (AND ONLY TWO) OF THE QUESTIONS IN EACH AREA.

OF THE TOTAL OF EIGHT QUESTIONS THAT ARE DONE, A MINIMUM OF TWO MUST B
FROM THE 600-LEVEL QUESTIONS.

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THE CATHOLIC UNIVERSITY OF AMERICA

Preliminary Examination--Physics Dept.

Thursday, March 20, and Friday, March 21, 2003

Room 133 - Hannan Hall

- **YOU MUST DO TWO QUESTIONS IN EACH OF THE AREAS:**

Thursday, March 20, 2003

Mechanics

Electricity & Magnetism

Friday, March 21, 2003

Thermodynamics

Modern Physics/Quantum Mechanics

- **DO EACH PROBLEM IN A SEPARATE BLUE BOOK**

- **PUT YOUR NAME ON EACH BLUE BOOK**

- **LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS**

  FOR EXAMPLE: Mechanics #1

CUA

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Department of Physics
200 Hannan Hall
Washington, D.C. 20064
202-319-5315
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Ph.D. Comprehensive Examination

Physics Department

Spring 2003

Thursday, March 20, and Friday, March 21, 2003

Room 133 - Hannan Hall

GENERAL INSTRUCTIONS:

This examination is divided into four sections as follows:

Thursday, March 20, 2003

9:00 a.m. - 12:00 Noon	Classical Mechanics - 2 questions
1:00 P.M. - 5:00 P.M.	E & M - 3 questions

Friday, March 21, 2003

9:00 a.m. - 12:00 Noon	Stat Mech. - 2 questions
1:00 P.M. - 5:00 P.M.	Quantum Mech. - 3 questions

DO EACH PROBLEM IN A SEPARATE BLUE BOOK

PUT YOUR NAME ON EACH BOOK

LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS
FOR EXAMPLE: Mechanics 600-1

A vertically launched rocket has an initial mass m_i, including fuel, at time $t = 0$. The rocket continues to follow a vertical path after launch. The rocket consumes its fuel supply at a constant rate $c = dm/dt$. The fuel is expelled at a constant velocity, v_m, relative to the rocket.
Show that the expression for the velocity of the rocket at some arbitrary time t is given by

$$u = v_m \ln \left[\frac{m_i}{m_i - ct} \right] - gt.$$

Hint: Consider the momentum of the rocket under the influence of gravity and the fuel at time t and at time $t + dt$.

A particle of mass m moves in one-dimension under a conservative force with potential energy

$$V(x) = \frac{cx}{x^2 + a^2}$$

where c and a are positive constants.

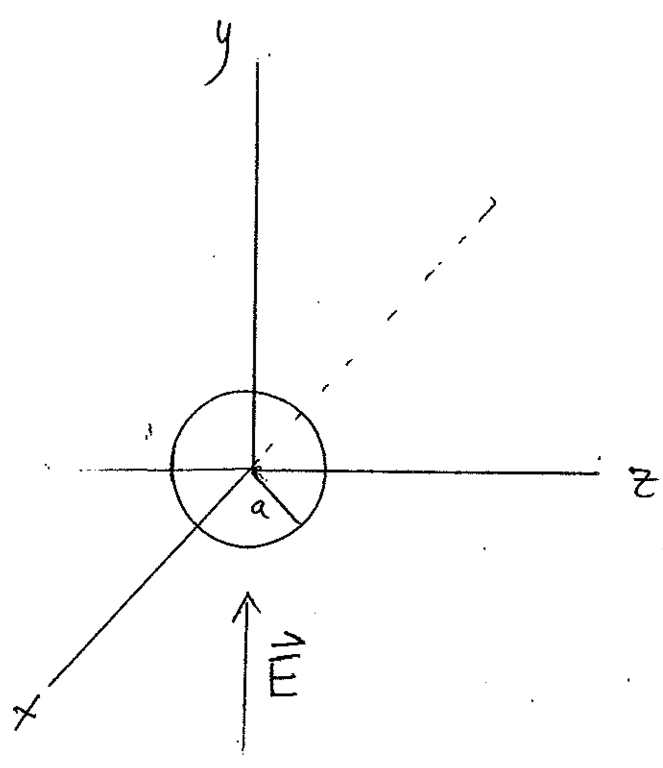
(a) Sketch the graph of $V(x)$ for $-\infty \leq x \leq +\infty$.
(b) Find the position of ***stable*** equilibrium and the period of small oscillations about it.
(c) If the particle starts out at the position of stable equilibrium with velocity v, find the range of values of v for which it:
  (i) Oscillates
  (ii) Escapes to $x = -\infty$
  (iii) Escapes to $x = +\infty$

1. A plane electromagnetic wave traveling in a vacuum is given by $\vec{E} = \hat{y}\, E_0 cos(kz - \omega t)$, where E_0 is real and $\hat{y}$ is a unit vector in the $+y$ direction. A one-turn circular loop of wire, of radius a, lies in the yz plane with its center at the origin (see drawing below):

a) Find all components of the magnetic field B associated with this wave.

b) Find the EMF induced in the loop as a function of time. Assume that the loop is small compared to the wavelength of the wave ($a << \lambda$).

c) Such a loop can be used as a "loop antenna" to detect electromagnetic waves (e.g. in an AM radio). What can you say about the best and worst orientations of such a "loop antenna" relative to the polarization and propogation directions of the incident wave?



Consider a conducting sphere of radius r = a immersed in a uniform electric field $\mathbf{E}_0$. The uniform electric field is produced by plane-parallel plate capacitor consisting of two very large plates of opposite charge. The plates are far away from the conducting sphere such that any charge distribution of the sphere does not affect that of the plates.

a.) Draw the electric field lines in the neighborhood of the conducting sphere.

b.) Using any method that you would prefer, derive an expression for the potential inside the capacitor that is valid both near the plates, and near, but still exterior to, the conducting sphere.

c.) Also show that the electric field components corresponding to this potential are:

$$E_r = E_0\left(1+\frac{2a^3}{r^3}\right)\cos(\theta) \quad \& \quad E_\theta = -E_0\left(1-\frac{a^3}{r^3}\right)\sin(\theta)$$

d.) What is the surface charge distribution on the conducting sphere?

e.) What is the charge distribution and electric field interior to the conducting sphere?

The tension τ along the axis of an ideal elastic cylinder is given by the equation of state:

$$\tau = aT\left[\frac{L}{L_0} - \frac{L_0^2}{L^2}\right]$$

where a is a constant, T is the absolute temperature, L is the length of the cylinder, and L_0 is the length at zero tension; L and L_0 are functions of T.

(a) The cylinder is stretched reversibly and isothermally from $L = L_0$ to $L = 2L_0$.

(i) Write out the differential forms for the work done by the system dW and the Helmholtz free energy dF.

(ii) Show that the Helmholtz free energy, $F(L,T)$, is:

$$F(L,T) = aT\left[\frac{L^2}{2L_0} + \frac{L_0^2}{L} - \frac{3L_0}{2}\right] + F(L_0,T)$$

(iii) Find the entropy change, ΔS.

(iv) Find the heat absorbed by the cylinder.

Note: Write your results to (ii) and (iii) in terms of a, T, L_0, and α_0, where α_0 is the thermal expansion coefficient at zero tension:

(b) If, instead, the length is changed adiabatically, the temperature of the cylinder changes.

$$\alpha_0 = \frac{1}{L_0}\frac{dL_0}{dT}.$$

Derive an expression for the "elastocaloric coefficient" $(\partial T/\partial L)_S$ in terms of a, T, L, L_0, α_0, and C_L, the heat capacity at constant length.

If the compression and rarefraction in producing a sound wave is an adiabatic process,

a.) Show that that the speed of sound, v, is

$$v = \sqrt{\frac{\gamma P}{\rho}},$$ where $\gamma = C_P/C_V$, P is pressure, and ρ is density.

Note: Consider an ideal gas, and assume that if there is compression or rarefraction, we must have a corresponding temperature change.

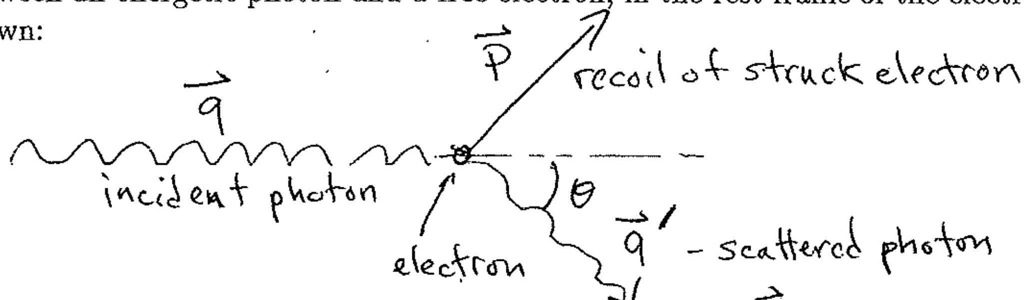
The speed of sound is also $v = \sqrt{\frac{B}{\rho}}$, where B is the adiabatic bulk modulus of the fluid,

or
$B = -V(\partial P / \partial V)_S$.

b.) If the gas is a monatomic gas, what are the values of C_V and C_P?

c.) What are the values of C_V and C_P for a diatomic gas?

1. Consider an atomic gas, containing some free electrons. Now, consider the collision between an energetic photon and a free electron, in the rest frame of the electron, as shown:



Note: the momentum of the electron after the collision is $\vec{p}$ and the momentum of the photon before the collision is $\vec{q}$ and, after the collision, is $\vec{q}\,'$.

a) using Conservation of Energy and Conservation of Momentum, show that change in the wavelength of the scattered photon is given as: $\lambda' - \lambda = (h/m_e c)(1 - \cos(\theta))$. [Note: frequency = c/λ].

b) In a Compton Scattering experiment, an incoming X-ray, of wavelength $\lambda = 5.53 \times 10^{-2}$ nm enters a gas and is scattered and deflected at an angle of 35^0. The fractional shift in wavelength of the scattered X-ray is found to be about 1% of its initial wavelength (an easily measurable shift). However, in this experiment some of the deflected X-rays seem not to have had their wavelengths shifted. Explain what might have occurred.

c) For what Kinetic Energy will a particle's de Broglie wavelength equal its Compton wavelength?

A particle of mass m is confined in a one-dimensional infinite square-well potential $V(x)$ with

$$V(x) = 0 \text{ for } 0 \le x \le a$$

$$V(x) = \infty \text{ otherwise.}$$

(a) Write out the time-independent Schrodinger equation and the stationary-state eigenfunctions and energies for this system.

(b) At time t = 0, the normalized wave function of the particle is

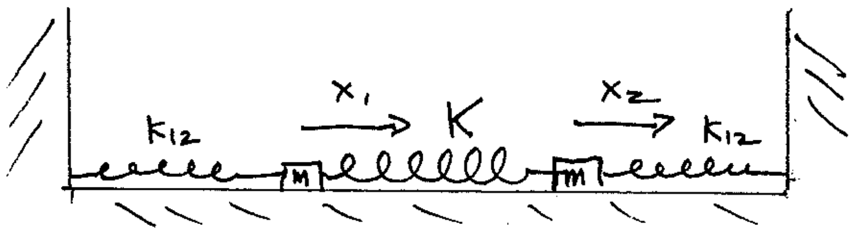
$$\psi(x) = \sqrt{\frac{8}{5a}}\left[1 + \cos(\pi x / a)\right]\sin(\pi x / a)$$

What is the average energy of the system?

(c) What is the wave function at a later time $t = t_o$?

(d) What is the probability that the particle is found in the region $0 \le x \le a/2$ at $t = t_o$?

1. Two identical masses slide horizontally without friction, but are subject to spring forces, as shown:



The displacements x_1 and x_2 are measured from equilibrium. Note that the center spring differs from the end springs.

a) Find the normal mode frequencies, ω_a and ω_b. Express your results in terms of the parameters $\omega_0 = \sqrt{k_{12}/m}$ and $\Omega_0 = \sqrt{K/m}$.

b) Show that $\eta_1 = x_1 - x_2$ and $\eta_2 = x_1 + x_2$ are the normal mode coordinates of the system.

c) Show that the Lagrangian is diagonal (no terms of the sort $q_i q_j$, where $i \neq j$) when expressed in normal mode coordinates.

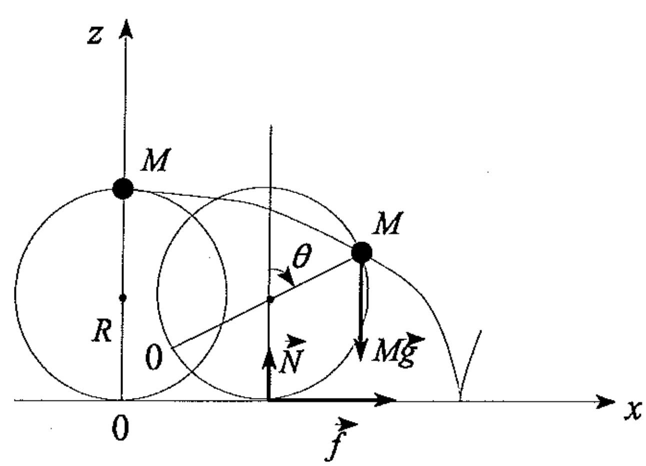
d) Find the Hamiltonian in terms of the normal mode coordinates.

A point mass M is attached to the rim of a massless cylinder of radius R, which rolls without slipping on a horizontal plane subject to vertical gravity $\vec{g}$. Initially the cylinder has a vanishing angular velocity $\dot{\theta}_0 \to 0^+$ and M is at its highest point. Find the subsequent angle θ_l at which the cylinder lifts itself up from the plane.

You may proceed as follows. The mass point M follows a cycloidal path with $x = R\theta + R\sin\theta$ and $z = R + R\cos\theta$ (see figure).

The floor exerts a horizontal force $\vec{f}$ on the cylinder, preventing it from slipping. The floor also exerts a vertical reaction force $\vec{N}$, which decreases to zero at the point where the cylinder lifts itself up from the floor.

(a) Balance the total torque of $\vec{f}$ and $\vec{N}$ with respect to the mass center, which coincides with the point mass M.
(b) Write the force equations for the acceleration components $\ddot{x}$ and $\ddot{z}$ of the mass center.
(c) Eliminating f and N from the three equations in (a) and (b), determine $\ddot{\theta}$ in terms of θ and $\dot{\theta}$, which provides the equation of motion for the Lagrangian coordinate θ.
(d) Write the equation of conservation of energy, and verify that by taking its time derivative, you obtain the same equation of motion for $\ddot{\theta}$ as you got in (c).
(e) Using (a), (b) and (c), express N in terms of θ and $\dot{\theta}$ only.
(f) Using (d) express N finally in terms of θ only.
(g) Determine θ_l at which N vanishes and the cylinder lifts up.



1. Consider a low-density plasma that consists of free electrons of mass m and charge $-e$. There are N charges per unit volume. Assume that the density is uniform and interactions between the charges can be neglected [imagine there is a background of fixed charges $q = +e$ so that the plasma is overall neutral. Such a background could be provided by ions with masses $m_i >> m$, hence their contribution to the conductivity would be negligible]. Electromagnetic waves (frequency ω, wave number k) are incident on the plasma.

a) Find the conductivity σ as a function of ω

b) Find the dispersion relation, i.e., find the relation between k and ω.

c) Find the index of refraction as a function of ω. The plasma frequency is defined by $\omega_p^2 \equiv Ne^2/m\epsilon$, where ϵ is the permittivity. What happens if $\omega < \omega_p$?

d) Now suppose there is an external magnetic field $\vec{\mathbf{B}}_o$. Consider plane waves travelling parallel to $\vec{\mathbf{B}}_o$. Show that the index of refraction is different for right- and left-circularly polarized waves.

Hints:

Assume that the magnetic field of the traveling wave is negligible compared to $\vec{\mathbf{B}}_o$. Let the external B field be $\vec{\mathbf{B}}_o = B_0\hat{\mathbf{z}}$; investigate the propogation of polarized waves traveling in the z-direction. Hence, you can start with the electric field :

$$\vec{\mathbf{E}} = E_0 e^{i(kz-\omega t)}[(1/\sqrt{2})\ (\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})], \tag{1}$$

where the + relation gives the polarization vector for the left-hand circularly polarized (LHCP) waves and the $-$ is for the right-hand (RHCP) ones.

To get started, define the polarization vectors:

$$\hat{\epsilon}_+ = (1/\sqrt{2})\ (\hat{\mathbf{x}} + i\hat{\mathbf{y}}) \tag{2}$$

$$\hat{\epsilon}_- = (1/\sqrt{2})\ (\hat{\mathbf{x}} - i\hat{\mathbf{y}}) \tag{3}$$

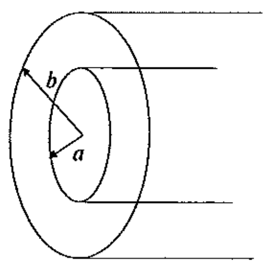
where $\hat{\epsilon}_+$ is for LHCP and $\hat{\epsilon}_-$ for RHCP. Note that: $\hat{\epsilon}_\pm \times \hat{\mathbf{z}} = \pm i\hat{\epsilon}_\pm$

Also, let the position of the charge be given by $\vec{\mathbf{x}} = x^+\hat{\epsilon}_+ + x^-\hat{\epsilon}_-$ (are there any forces in the z-direction?), and assume the time dependence of x as $e^{i\omega t}$.

Finally, write your final expression in terms of the cyclotron frequency: $\omega_c = eB_o/m$.

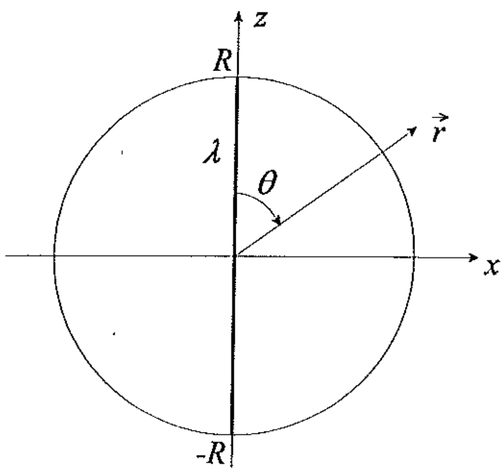
Consider two very long thin-walled concentric cylinders of radii a and b, such that $a < b$. the inner cylinder is maintained at a constant potential $\Phi(a) = V_a$, the outer at $\Phi(b) = V_b$. The space between the cylinders contains a space charge of density $\rho = kr$, where k is a constant and r is the distance from the central axis of both cylinders.

a.) What is the distribution of the potential, $\Phi(r)$, between the cylinders?
b.) What is the surface charge density on the two cylinders?



MS/PhD Comprehensive Examination
Spring 2003
Electromagnetism 600-3

Consider a segment of linear charge density $\lambda = Q / 2R$ placed along the z-axis with its endpoints at $z = \pm R$ (see figure).



(a) Determine the electrostatic potential $\Phi(r, \theta)$ in the region where $r > R$, *i.e.*, outside the circumscribing sphere of radius R having the segment charge as its diameter. Here $\vec{r}$ is the observation point of the potential, forming a polar angle θ with the z-axis.

(b) Determine $\Phi(r, \theta)$ for $r \le R$, *i.e.*, inside the circumscribing sphere.

HINT: In both cases provide $\Phi(r, \theta)$ as a series expansion in Legendre polynomials, recalling that

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{l=1}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma).$$

MS/PhD Comprehensive Examination
Spring 2003
Statistical Mechanics 600-1

Consider a system of N quantum spin-½ magnetic moments in a magnetic field $\vec{B}$. Each magnetic moment has only two microstates with energies $-\varepsilon$ and $+\varepsilon$, where $\varepsilon = \mu_B B$. If a magnetic moment is aligned with $\vec{B}$, it has energy $= -\varepsilon$ and there are N_+ such aligned moments. If a magnetic moment is anti-aligned with $\vec{B}$, it has energy $+\varepsilon$ and there are N_- such anti-aligned moments. Now assume that $N = N_+ + N_-$ and $E = -\varepsilon N_+ + \varepsilon N_- \equiv -\varepsilon m$, which is the total energy of the system , are all fixed, corresponding to a microcanonical ensemble description of the system of magnetic dipoles.

(a) Determine the number of microscopic states $\Omega\ (N, E) = \Omega\ (N_+, N_-)$ of the system of magnetic dipoles for a given n and E, or, equivalently, for given N_+ and N_-, with $N_\pm = \frac{1}{2}(N \pm m)$.

(b) Determine the entropy $S\ (N, E)$ of the system of magnetic dipoles in the thermodynamic limit.

(c) Determine the equation of state that expresses $1/T$ as a function of E and N. Invert it to solve for E as a function of N and T.

(d) Still in terms of given N, N_+, and N_-, what are the microcanonical probabilities p_+ and p_- that any given magnetic dipole is aligned or anti-aligned with $\vec{B}$?

(e) Now determine the canonical probabilities $p_+(\beta)$ and $p_-(\beta)$ at given $\beta = 1/kT$ that any given magnetic dipole is either aligned or anti-aligned with $\vec{B}$.

(f) Equate the microcanonical and canonical probability ratio

$$\frac{p_+}{p_-} = \frac{p_+(\beta)}{p_-(\beta)}$$

to obtain the relation between E and T, and compare it with that in part (c).

(a) Consider an ideal gas of N indistinguishable particles of mass m confined to a volume V at a temperature T. Using the classical approximation for the canonical partition function, z, calculate the chemical potential μ of the gas.

(b) A gas of n indistinguishable classical particles, also of mass m, is absorbed onto a surface of area A forming a two-dimensional ideal gas at temperature T. The energy of an absorbed particle is $\varepsilon = (p^2/2m) - \varepsilon_0$, where the momentum $\boldsymbol{p}$ has components p_x and p_y and ε_0 is the surface binding energy. Calculate the classical canonical partition function z_s and the chemical potential μ_s of this gas.

(c) If at temperature T the particles on the surface and in the surrounding three-dimensional gas are in equilibrium, find the mean number of particles absorbed per unit area when the pressure of the three-dimensional gas is P.

(d) Now consider the two-dimensional gas as a system that can exchange particles with the three-dimensional gas. Obtain the grand canonical partition function for the two-dimensional gas by considering the number of configurations for distributing n identical particles on K surface sites. Hence, obtain the mean fraction of surface sites occupied θ in terms of the pressure of the three-dimensional gas.

A particle of mass m is confined to move on a circle of radius a but is otherwise free. A perturbing potential, $H' = A\sin\theta\cos\theta$, is applied, where θ is the angular position on the circle.

(a) Write out the Schrodinger equation and find the unperturbed eigenfunctions and energies for this system (i.e., when $H'=0$).

(b) Find the first-order corrections to the energies of the two lowest states with $E \neq 0$ for this system when the perturbation is present.

(c) Find the "good" zeroth-order eigenfunctions for these two states.

(d) Find the second-order corrections to the.energies of the two lowest states with $E \neq 0$ for this system when the perturbation is present.

Consider the diatomic hydrogen molecule, H_2, and the diatomic deuterium molecule, D_2. We are concerned with the degrees of freedom associated with the rigid rotations of these linear molecules and with the nuclear spins of their constituent H and D atoms.

(a) The Hamiltonian of a rigid rotator is $\hat{H}_{rot} = \hat{L}^2/2I$. What is the spectrum of eigenvalues? What are the degeneracies of the energy levels? What is the symmetry of the eigenfunctions under inversion, bringing $\theta \to \pi - \theta$ and $\varphi \to \varphi + \pi$?

(b) In the H_2 molecule, the nuclei of the two H-atoms are spin-½ fermions. What are the eigenvalues of the total nuclear spin square angular momentum $\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$? What are the corresponding degeneracies? What is the symmetry of the $\hat{S}^2$- eigenstates under exchange of the two nuclei?

(c) In the D_2 molecule, the nuclei of the two D-atoms are spin-1 bosons. What are the eigenvalues of the total nuclear spin square angular momentum $\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$? What are the corresponding degeneracies? What is the symmetry of the $\hat{S}^2$- eigenstates under exchange of the two nuclei?

(d) A rotation of a linear molecule that brings $\theta \to \pi - \theta$ and $\varphi \to \varphi + \pi$ amounts to both an inversion on the rotational eigenfunctions and an exchange of the molecular nuclei. In homonuclear molecules, the Pauli Principle will thus restrict the total rotational-spin eigenstates that can be physically realized. For the H_2-molecule, which spin-eigenstates can be associated with which rotational eigenfunctions?
Answer the same question for the D_2-molecule.

A particle of mass m moves in a one-dimensional potential $U(x) = -U_0 e^{-\alpha x^2}$, where $U_0 > 0$ and $\alpha > 0$. Consider a variational wavefunction $\psi = e^{-\beta x^2}$, where β is a variational parameter.

(a) Calculate $\langle \psi | \psi \rangle$, $\langle \psi | \hat{T} | \psi \rangle$ and $\langle \psi | \hat{U} | \psi \rangle$, where

$$\hat{T} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$$

is the kinetic-energy operator. Then obtain

$$E(\beta) = \frac{\langle \psi | \hat{H} | \psi \rangle}{\langle \psi | \psi \rangle},$$

where $\hat{H} = \hat{T} + \hat{U}$ is the Hamiltonian.

(b) Derive the equation for β that minimizes $E(\beta)$.

(c) Now assume that $U_0 \ll \hbar^2 \alpha / m$, and, correspondingly $\beta_{min} \ll \alpha$ in the equation for β that minimizes $E(\beta)$. Solve the equation under these conditions, thus determining β_{min} and $E(\beta_{min})$.

(d) What is the relation between $E(\beta_{min})$ and the true ground state energy for $U(x)$? Then, argue that $U(x)$ admits always at least one bound state.