CUA

THE CATHOLIC UNIVERSITY OF AMERICA

Department of Physics
200 Hannan Hall
Washington, D.C. 20064
202-319-5315
Fax 202-319-4448

Ph.D. Comprehensive Examination

Physics Department

Spring 2002

Thursday, March 14, and March 15, 2002

Room 133 - Hannan Hall

GENERAL INSTRUCTIONS:

This examination is divided into four sections as follows:

Thursday, March 14, 2002	
9:00 a.m. - 12:00 Noon	Classical Mechanics - 2 questions
1:00 P.M. - 5:00 P.M.	E & M - 3 questions
Friday, March 15, 2002	
9:00 a.m. - 12:00 Noon	Stat Mech. - 2 questions
1:00 P.M. - 5:00 P.M.	Quantum Mech. - 3 questions

DO EACH PROBLEM IN A SEPARATE BLUE BOOK

PUT YOUR NAME ON EACH BOOK

LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS
FOR EXAMPLE: Mechanics 600-1

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MS Comprehensive Examination
Physics Department

Spring 2002

Thursday, March 14, and Friday March 15, 2002

Room 133 - Hannan Hall

GENERAL INSTRUCTIONS:

This examination is divided into four sections as follows:

Thursday, March 14, 2002	
9:00 a.m. - 12:00 Noon	Classical Mechanics - 2 questions
1:00 P.M. - 5:00 P.M.	E & M - 2 questions
Friday, March 15, 2002	
9:00 a.m. - 12:00 Noon	Thermodynamics/Stat. Mech. - 2 questions
1:00 P.M. - 5:00 P.M.	Modern Physics/Quantum Mech. - 2 questions

DO EACH PROBLEM IN A SEPARATE BLUE BOOK

PUT YOUR NAME ON EACH BOOK

LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS
FOR EXAMPLE: Mechanics 600-1 or Electromagnetism 500-2

YOU MUST DO TWO (AND ONLY TWO) OF THE QUESTIONS IN EACH AREA.

OF THE TOTAL OF EIGHT QUESTIONS THAT ARE DONE, A MINIMUM OF TWO MUST BE FROM THE 600-LEVEL QUESTIONS.

Ph.D/MS/PRELIM. Comprehensive Examination

Spring 2002

Thursday, March 14, and Friday, March 15, 2002

THE WRITTEN EXAM WILL BE HELD IN ROOM 133 HANNAN HALL

THE EXAM WILL BEGIN AT 9:00 AM ON March 14, AND CONTINUE THRU 5:00 PM FRIDAY, March 15, 2002.

THE ORAL PORTION OF THE PH.D. COMPREHENSIVE EXAM WILL BE THE FOLLOWING WEEK.

PLEASE CALL (202) 319-5315) OR STOP BY THE PHYSICS DEPARTMENT OFFICE ON FRIDAY TO INQUIRE ABOUT YOUR SCHEDULED ORAL EXAM.

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RULES FOR THE PHYSICS COMPREHENSIVE EXAMINATION

Spring 2002

In order to assure an equitable and fair Comprehensive Examination for all students the following rules will apply to the examination administered by the Department of Physics.

1. The examination will be closed book. Students will not be permitted to bring to the examination room materials such as notes, reference books, calculators, or other aids of any form. The Physics Department will supply copies of the *CRC Mathematical Handbook,* Schaum's *Mathematical Handbook, Table of Functions* by Jahnke and Emde, and the NBS *Handbook of Mathematical* Functions, for the examinees' use should they feel the need for these references during the examination.

   Students who want to use their own mathematical tables and/or handbooks must get permission from the Chairman at least two days before the beginning of the comprehensive. These tables and/or handbooks will remain under the control of the department until after the comp exam but will be available to their owners during the exam.

2. If needed, students will be allowed a short break, not to exceed 5 minutes, to use the restroom facilities during the examination period. While absent from the examination room students may not consult material or other sources of information concerning any matter pertaining to the examination. Students who leave the examination room will turn in all of their examination materials to the proctor before leaving the room. Examination papers will be returned to the students when they reenter the examination room. Students who fail to return within 5 minutes will be considered to have completed that portion of the examination.

<u>Only one student will be permitted to leave the examination room at a time.</u>

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Prelim./MS/Ph.D. Comprehensive Exam

Spring 2002

Proctoring Schedule

Room 133 - Hannan Hall

Thurs., March 14, 2002	9:00 - 10:30 - F. Klein 10:30 - noon - I. Pegg
	1:00-3:00 pm - P. Macedo 3:00-5:00 pm - R. Selinger
Fri., March 15, 2002	9:00-10:30 - F. Bruhweiler/S. Kraemer 10:30-12:00 noon - F. Bruhweiler/S. Kraemer
	1:00-3:00 pm - L. Resca 3:00-5:00 pm - C.J. Montrose

a) An object of mass m is projected vertically upwards from ground level with an initial velocity v_0. What is the maximum height h reached by the object if air resistance is neglected.

b) Now consider the case of free-fall where air resistance imparts a force $F = -bv^2$ on the object, where b is a constant. Obtain an expression for the terminal velocity v_t that is approached in free-fall.

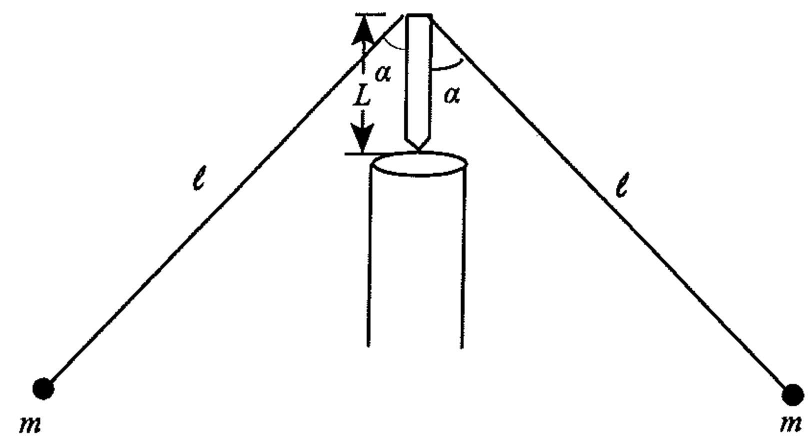
c) If the force of air resistance ($F = -bv^2$) is included in part (a), show that the maximum height reached by the object is given by:

$$h' = \frac{v_t^2}{2g} \ln\left(1 + \frac{v_o^2}{v_t^2}\right)$$

[Hint: Consider writing the equation of motion in terms of v and integrating by separation of variables.]

d) Calculate the work done by air resistance in part (c).

A "teeter toy," shown in the diagram, consists of two identical masses m on the ends of rigid rods; the other ends of the rods are connected to a pin. The rods and the pin lie in a common plane. The rods are each of length l and each makes an angle α with the pin. The pin rests on a fixed pedestal, as shown.



Consider only rocking motions of the teeter toy in the plane of the diagram such that the <u>plane of the toy remains vertical</u>. Measure the displacement by the angle θ that the pin makes with the vertical. Take the zero of gravitational potential energy to be the point at which the pin meets the pedestal. Assume that the masses of the rods and the pin are negligible compared to m.

a) Show that the potential energy of the toy when it is displaced by an angle θ from the vertical is given by:

$$V = 2mg\,(L - l\cos\alpha)\cos\theta\,.$$

b) Find the equilibrium value of θ from this potential energy.

c) Find the condition such that this equilibrium position is stable.

M. S. Comprehensive Exam
Spring, 2002
E & M 500-1

Two parallel infinite straight wires carry steady currents I_1 and I_2, respectively.

a) Using Ampere's law, determine the magnetic field $\vec{B}_{1 \to 2}$ that the first wire produces at the site of the second wire.

b) Using the Lorentz-force law, determine the force per unit length that the first wire exerts on the second wire. In which cases is the force attractive or repulsive?

A line charge exists on the z-axis between $z = a$ and $z = -a$. The line charge density, λ, varies with z and is given by:

$$\lambda = \lambda_0 \quad \text{for} \quad 0 < z < a$$
$$\lambda = -\lambda_0 \quad \text{for} \quad -a < z < 0.$$

For distances large compared to a:

a) Determine the monopole contribution to the potential caused by this charge distribution.

b) Calculate the dipole moment of this charge distribution.

c) What is the dipole contribution to the potential?

d) If an electric field given by $\quad E = \hat{i}\, E_o$

is present, what is the net force on the charge distribution?

e) If an electric field given by $\quad E = \hat{k}\, A\, z$

(where A is a constant) is present, what is the net force on the charge distribution?

M. S. Comprehensive Exam
Spring, 2002
Thermodynamics 500-1

A gas of initial volume V_1 is forced from Region 1 to Region 2 through a porous plug ("throttled") where it undergoes adiabatic expansion (see the diagram below). Pistons A and B are moved in such a way that the pressures p_1 and p_2 remain constant. The initial volume of Region 1 is V_1 and the final volume is zero. The initial volume of Region 2 is zero and the final volume is V_2. The gas is not necessarily ideal.

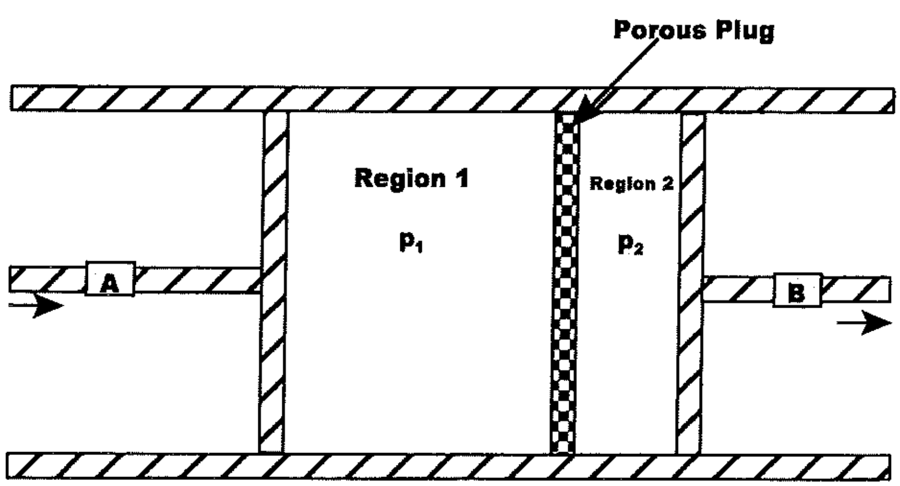
a) By considering the work done by the gas in this process, show that the initial and final enthalpies of the gas are equal.

b) Since from part (a) $dH=0$, use this result to show that for small pressure differences $\Delta p = p_2 - p_1$ the temperature difference between the two regions is given by

$$\Delta T = \frac{V}{C_p}(T\alpha - 1)\Delta p$$

where $\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$ *and* $C_p = \left(\frac{\partial U}{\partial T}\right)_p$.

c) Using this result, determine whether such a process can be used to cool an ideal gas.



M. S. Comprehensive Exam
Spring, 2002
Thermodynamics 500-2

Consider a heat engine that is operating in a Carnot cycle.

a) Plot the cycle on the pressure vs. volume graph.

b) Identify which properties are held constant in each part of the cycle.

c) Identify when heat flows in or out of the system.

d) Identify when work is done by or on the system.

e) What is the area contained by the cycle in the pressure vs. volume graph?

f) What is the change in entropy for each part of the cycle and what is the additional condition that makes the cycle a "Carnot" cycle?

g) What is the efficiency of the Carnot engine as a function of reservoir temperatures?

h) If the engine is used as a refrigerator, what is the efficiency as a function of input and output temperatures?

M. S. Comprehensive Exam
Spring, 2002
Quantum Mechanics 500-1

Consider a particle of mass m in an attractive central potential, $V(r) = -e^2/r^\beta$, where $\beta > 0$. Assume that the particle is in the state of lowest energy E, and its wavefunction is localized in a spherical region of uncertainty radius $\Delta r \sim a$.

a) Using the uncertainty principle, derive the following estimate for the total energy:

$$E(a) = \frac{\hbar^2}{2ma^2} - \frac{e^2}{a^\beta}.$$

b) Plot $E(a)$ as a function of a; plot the case $0 < \beta < 2$ and the case $\beta > 2$ separately.

d) Find the radius a_0 which minimizes $E(a)$ for $0 < \beta < 2$. For the Coulomb case, $\beta = 1$, compare your result for a_0 with the Bohr radius of the hydrogen atom

$$a_B = \frac{\hbar^2}{me^2}.$$

Calculate the corresponding $E(a_0)$ and compare it to the exact hydrogen atom ground state energy.

e) What happens in the $\beta > 2$ case? Is the system stable, or does the particle collapse onto the center of attraction? What ultimately is the physical principle that prevents that from happening in the $\beta < 2$ case?

Consider the "delta-function" potential

$$V(x) = -V_0 \delta(x) \qquad (V_0 > 0)$$

in the one-dimensional Schrödinger equation. In this problem, you are asked to show that:
(i) For $E < 0$ there exists only one bound state localized by this potential (parts a-c); and
(ii) For $E > 0$ a current of particles, arriving from the left (at $x = -\infty$), produces a scattered wave, emanating from the wall in both the backward and forward directions (parts d-f).

a) For $E < 0$, adjust the solution $\psi(x)$ of the Schrödinger equation for $x > 0$ and $x < 0$ to satisfy the boundary conditions for $x \to \pm\infty$, respectively.

b) Evaluate the behavior of $\psi(x)$ at $x = 0$. Note that $\psi(x)$ must be continuous at the origin. [Hint: Integrate the Schrödinger equation over the interval $(-\varepsilon, +\varepsilon)$ and consider the limit $\varepsilon \to 0$.]

c) Find the energy eigenvalue for $E < 0$.

d) For $E > 0$, proceed as in (a) and (b) to find the solutions of the Schrödinger equation for $x > 0$ and $x < 0$ for an incoming plane wave from the left.

e) For $E > 0$, find a relation between V_0 and both the reflected and transmitted amplitudes of $\psi(x)$, and verify the equation of continuity, which states that the sum of the reflected and transmitted intensities must equal the incident intensity.

f) Show that we approach total reflection in the limit $V_0 \to \infty$ and that the back-scattered intensity becomes inversely proportional to the particle energy for small V_0.

Ph.D. Comprehensive Exam
Spring, 2002
Mechanics 600-1

Consider a "loaded" string, that is, an elastic string on which identical particles are attached at regular intervals such that:

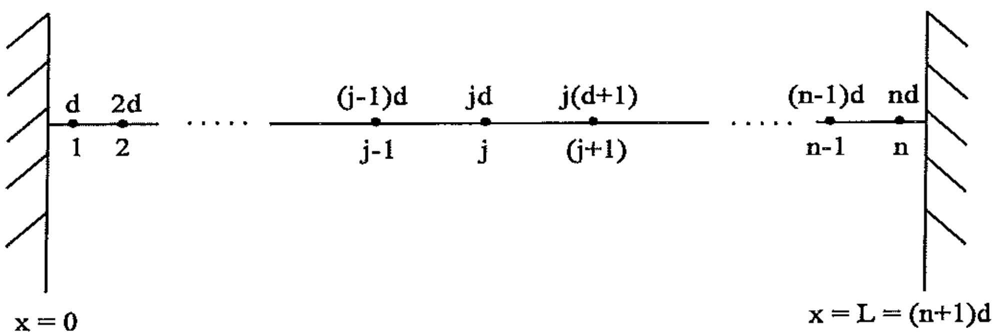
(i) The mass of each of the n particles is m.

(ii) The spacing between the masses is d.

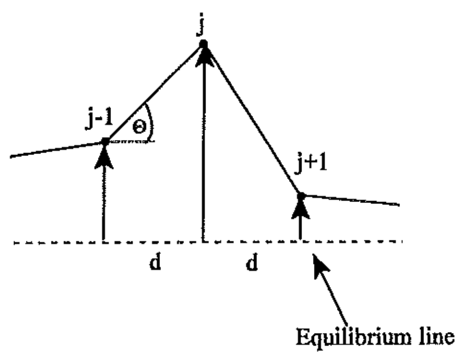
(iii) The length of the string $L = (n + 1)d$ and the string is fixed at $x = 0$, and $x = (n + 1)d$.

(iv) The string is taut and L is so long that the boundaries can be ignored.

(v) The tension in the string is τ and gravity can be ignored.



Consider small vertical displacements say of particles $j-1, j, j+1$, such that τ is approximately constant and equal to its value at equilibrium, as shown below.
Assume that θ is small such that $\sin\theta \approx \tan\theta$.



a) What is the restoring force on particle j?

b) Using the expression for force, what is the equation of motion for particle j?

c) From the equation of motion, derive the one-dimensional wave equation in the limit of $d \to 0$ and $m/d \to \rho$.

d) What is the speed of the wave?

Ph.D. Comprehensive Exam
Spring, 2002
Mechanics 600-2

Consider a canonical transformation produced by a generating function $F(p, Q, t) = -p\, Q \cos(\omega t)$. Applying Hamilton's principle with varied paths having $\delta p = \delta Q = 0$ at the extrema yields

$$-\dot{p}q - H = P\dot{Q} - K + \frac{dF}{dt}.$$

a) From this principle, derive the equations $q = -\left(\frac{\partial F}{\partial p}\right)_{Q,t}$ and $P = -\left(\frac{\partial F}{\partial Q}\right)_{p,t}$ and the corresponding transformed canonical variables (Q, P) in terms of the original ones (q, p).

Now consider the Hamiltonian $H(q,p,t) = \dfrac{p^2 \cos^2(\omega t)}{2m} + \dfrac{m\omega^2 q^2}{2\cos^2(\omega t)} - \omega p q\, tg(\omega t)$

b) Use the previous canonical transformation to show that the transformed Hamiltonian

$$K(Q,P) = H(q,p,t) + \left(\frac{\partial F}{\partial t}\right)_{p,Q}$$

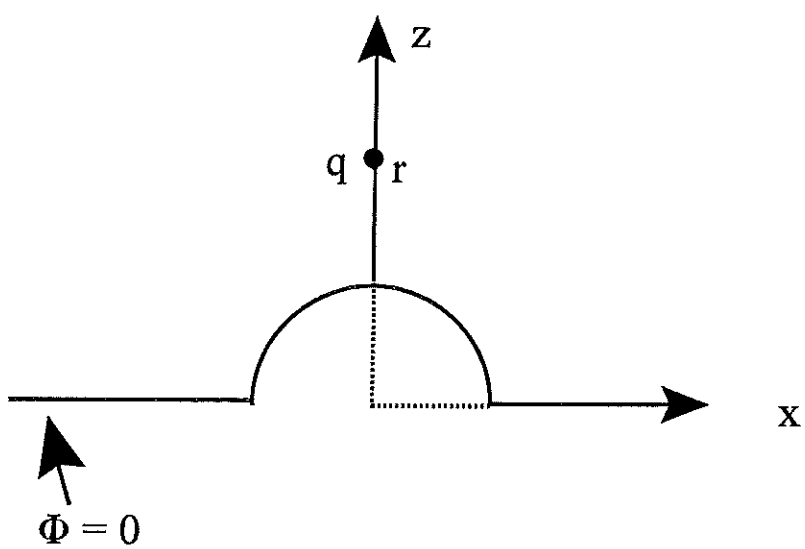
is the harmonic oscillator Hamiltonian.

c) Solve Hamilton's Equations for $K(Q, P)$, thus determining $Q(t)$ and $P(t)$, in terms of two initial arbitrary constants.

d) Determine the corresponding solution $q(t)$ and $p(t)$ to Hamilton's Equations for $H(q, p, t)$.

a) Using the image charge method, determine the force on a point charge q at a distance r from the center of a grounded conducting sphere of radius $a < r$.

b) Now consider a conductor at potential $\Phi = 0$, which has the shape of an infinite conducting plane in the x, y-plane, except for a hemispherical bump of radius a, as shown below. The center of the hemispherical bump is located at the origin $x = y = z = 0$, and the top of the bump is at $z = +a$. What is the force on a point charge q if it is placed on the z-axis at $z = r > a$?



[Note: The diagram shows a cross-section through the hemisphere in the x, z-plane.]

Ph.D. Comprehensive Exam
Spring, 2002
E& M 600-2

Consider a uniformly magnetized sphere of radius R and constant magnetization $\vec{M}_0 = M_0\hat{z}$, along the polar z-axis.

a) Show that the magnetic field $\vec{H}$ can be derived from a magnetic scalar potential as $\vec{H} = -\vec{\nabla}\Phi_M$, and Φ_M must be continuous at $r = R$.

b) Since the magnetic induction is $\vec{B} = \vec{H} + 4\pi\vec{\mathrm{M}}$, show that $\nabla^2\Phi_M = 0$ for both $r < R$ and $r > R$.

c) Find $\Phi_M(r, \theta)$ for both $r < R$ and $r > R$, based on the previous conditions and results, and further requiring continuity of $B_\perp = B_r$ at $r = R$.

d) Determine $\vec{B}$ and $\vec{H}$ for $r \leq R$, in terms of $\vec{M}_0$.

A particle of mass m and charge e moves in a constant uniform magnetic field $\vec{B}$, subject to the Lorentz force equation of motion

$$\vec{F} = \frac{e}{c}\vec{v} \times \vec{B} = \frac{d\vec{p}}{dt}.$$

a) In the non-relativistic limit, show that the corresponding energy rate-of-change is $\vec{v} \bullet \vec{F} = \dfrac{dE}{dt}$. Is the energy E conserved?

b) Correspondingly, show that the projection of the motion in a plane perpendicular to $\vec{B}$ is circular. Determine the angular velocity ω in $v_\perp = \omega a$, where a is the radius of the circle.

c) Determine the time evolution of the velocity component $v_{/\!/}$ parallel to $\vec{B}$.

In the more general relativistic treatment, both the force and the energy equations still apply, but with $\vec{p} = \gamma m \vec{v}$ and $E = \gamma mc^2$, where $\gamma = \dfrac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}$.

d) Assume that $v_{/\!/} = 0$. Find the general relativistic expression for the angular velocity ω.

e) Determine the general relativistic expression for $v_\perp = \omega a$ as a function of the orbit radius a. Determine the limits of that expression for $v_\perp$ when $a << \dfrac{mc^2}{eB}$ and $a >> \dfrac{mc^2}{eB}$. Compare these results with the expected non-relativistic and ultra-relativistic limits.

Consider a one-dimensional system of N classical non-interacting particles subject to an external potential

$$V(z) = V_0 \log\left[1 + \frac{|z|}{L}\right],$$

where V_0 and L are constants.

a) Plot $V(z)$.

b) Calculate the single-particle partition function

$$Q_1(\beta) = \frac{1}{h}\int_{-\infty}^{+\infty} dz \int_{-\infty}^{+\infty} dp_z e^{-\beta H}$$

where $H(z, p_z) = \frac{p_z^2}{2m} + V(z)$ *and* $\beta = \frac{1}{kT}$. For which values of kT is $0 \leq Q_1 < +\infty$?

[Hint: The change of variable $\zeta = 1 + z/L$ may be helpful in evaluating an integral.]

c) Calculate the average energy of the system $U = -NkT\frac{\partial}{\partial\beta}(\log Q_1)$.

d) Referring back to your plot in part (a), What is the largest distance z_ε that a particle of energy ε can reach?

(e) Is the particle confined by the potential $V(z)$ when $\varepsilon \to +\infty$? Discuss why the system cannot be in thermodynamic equilibrium for $kT \geq V_0$. Compare this system with the situation of $V(z) = 0$ for all $-\infty < z < +\infty$, where there are no walls to confine the particles at all.

Consider a monoatomic isotropic two-dimensional square lattice of N atoms with lattice spacing a. Assume Debye's model, with dispersion relation

$$\omega = c\sqrt{k_x^2 + k_y^2}\ .$$

a) Determine the corresponding density of states $g(\omega)d\omega$.

b) Determine the Debye cut-off frequency ω_D.

c) Determine the exact expression for the average energy U.

d) Determine U in the $kT >> \hbar\omega_D$ limit, and compare your result with that of the classical equipartition theorem.

e) Determine U in the $kT << \hbar\omega_D$ limit.

Ph.D. Comprehensive Exam
Spring, 2002
Quantum Mechanics 600-1

A particle of mass m is in a one dimensional harmonic oscillator potential

$$V(x) = \frac{1}{2} m\omega^2 x^2.$$

The raising and lowering operators are

$$a_{\pm} = \frac{1}{\sqrt{2m}}\left(\frac{\hbar}{i}\frac{d}{dx} \pm im\omega x\right),$$

such that $a_+|n\rangle = c_+|n+1\rangle$ and $a_-|n\rangle = c_-|n-1\rangle$

a) Without using any knowledge of the specific form of the harmonic oscillator eigenfunctions except for their orthonormality, show that

$c_+ = \sqrt{(n+1)\hbar\omega}$ and $c_- = \sqrt{n\hbar\omega}$

[Hint: Note that $\langle a_+\psi|a_+\psi\rangle = \langle\psi|a_-a_+\psi\rangle$ *and* $\langle a_-\psi|a_-\psi\rangle = \langle\psi|a_+a_-\psi\rangle$.]

b) Use these results to obtain an expression for the matrix elements $\langle m|x|n\rangle$.

c) If a perturbation of $H' = -\alpha x$ is now applied, show that there is no change in the energies to first-order and calculate the second-order corrections.

d) With the perturbation applied, the Schrödinger equation can be solved exactly with the change of variables

$$X = x - \frac{\alpha}{m\omega^2}.$$

Find the exact energies and show that they are consistent with the results of part (c).

Due to spin, an electron carries a magnetic moment: $\vec{\mu} = \gamma\ \vec{S}$, where

$\vec{S} = \frac{\hbar}{2}\vec{\sigma}$, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\gamma = -\frac{e}{mc} < 0$ is the quantum gyromagnetic ratio.

Given the eigenstates

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

of S_z, suppose that the spin is initially in the state

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}\left[|+\rangle + |-\rangle\right].$$

a) Of which spin-operator is $|\Psi(0)\rangle$ an eigenstate, and with which eigenvalue? In which direction is $|\Psi(0)\rangle$ pointing?

Now, for $t \geq 0$, a uniform magnetic field $\vec{B}$ is applied in the z-direction, producing an interaction Hamiltonian for the spin

$$H = -\vec{\mu} \bullet \vec{B} = \mu_B B\, \sigma_z, \text{ where } \mu_B = -\frac{\hbar}{2}\gamma = \frac{\hbar e}{2mc} \text{ is the Bohr magneton.}$$

b) Determine the exact time evolution $|\Psi(t)\rangle$ of our initial $|\Psi(0)\rangle$.

c) Determine the period T after which $|\Psi(T)\rangle$ returns to be a spin eigenstate pointing in the positive x-direction. What is the corresponding angular frequency $\omega = 2\pi/T$ of the precession? How does that ω compare with the classical Larmor frequency?
[Hint: Extract from $|\Psi(t)\rangle$ an overall inessential phase factor and concentrate on the Bohr frequency of transition.]

d) After which time is the spin pointing in the positive y-direction? Prove that by considering the eigenstates of S_y.

e) Determine the average values $\langle S_i\rangle_t = \langle\Psi(t)|S_i|\Psi(t)\rangle$ for all three spin components $i = x, y, z$.

Ph.D. Comprehensive Exam
Spring, 2002
Quantum Mechanics 600-3

A point-charge rigid rotator of mass m, moment of inertia $I = mr_0^2$, and charge e is constrained to rotate in the x-y plane. The corresponding free-rotation Hamiltonian is $\hat{H}_0 = \frac{1}{2I}\hat{L}_z^2$, where $\hat{L}_z^2$ is the z-component of the angular momentum operator, which is given by $\frac{\hbar}{i}\frac{\partial}{\partial\phi}$ in the (r,ϕ) coordinate representation.

a) Find the eigenvalues and eigenfunctions of $\hat{L}_z$. Are the eigenvalues quantized? Are any eigenvalues degenerate?

b) Find the energy eigenvalues E_m^0 and eigenfunctions $u_m^0(\phi)$ of $\hat{H}_0$, and compare them with those of $\hat{L}_z$. Which energy eigenvalues are degenerate?

Now a magnetic field $\vec{B}$ perpendicular to the x-y plane is introduced, causing an interaction Hamiltonian $\hat{H}_1 = -\frac{e\vec{B}}{2mc}\hat{L}_z$.

c) Find the energy eigenvalues and eigenfunctions of the complete Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_1$. Are the degeneracies typically lifted? For which particular values of B can "accidental" degeneracies be introduced?