Ph.D. Comprehensive Examination

Physics Department

Fall 2013

Thursday, October 24 and Friday, October 25, 2013

GENERAL INSTRUCTIONS:

This examination is divided into four sections as follows:

Thursday. October 24, 2013		
9:00 am. .12:00 Noon	Classical Mechanics	- 2 questions
1:00P.M. .5:00 P.M.	Electricity & Magnetism	- 3 questions
Friday. October 25. 2013		
9:00 a.m. .12:00 Noon	Statistical Mechanics	- 2 questions
1:00P.M. .5:00 P.M.	Quantum Mechanics	- 3 questions

In each of the four subject areas, you may answer the **500-level question** in place of a 600-level question.

DO EACH PROBLEM IN A SEPARATE BLUE BOOK

PUT YOUR NAME ON EACH BOOK

LABEL EACH BOOK WITH THE CORRECT PROBLEM NUMBER,
FOR EXAMPLE: "Mechanics 600-1"

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RULES FOR THE PHYSICS COMPREHENSIVE EXAMINATION
Fall 2013 (October 24 & 25, 2013)

In order to assure an equitable and fair Comprehensive Examination for all students, the following rules will apply to the examination administered by the Department of Physics.

1. The examination will be **closed book**. Students will not be permitted to bring to the examination room materials such as notes, reference books, calculators, or other aids of any form. The Physics Department will supply copies of the *CRC Mathematical Handbook*, Schaum's *Mathematical Handbook*, the *Table of Functions* by Jahnke and Emde, and the *NBS Handbook of Mathematical Functions,* for use during the examination.

   Students who want to use their own mathematical tables and/or handbooks must get permission from the Chairman prior to the beginning of the comprehensive examination. These tables and/or handbooks will remain under the control of the Department until after the examination is completed, but will be available to their owners during the examination period.

   The Physics Department will supply calculators for use during the examination.

2. If needed, students will be allowed a short break, not to exceed ten minutes, during the examination period. While absent from the examination room students may not consult material or other sources of information concerning any matter pertaining to the examination. Students who leave the examination room will turn in all of their examination material to the proctor before leaving the room. Examination papers will be returned to the students when they reenter the examination room. Students who fail to return within ten minutes will be considered to have completed that portion of the examination. Only one student will be permitted to leave the examination room at a time.

3. Students are not permitted to bring any electronic devices (including cell phones, PDAs and calculators) into the examination room. Any such devices must be handed over to the proctor for the duration of the exam session.

In a space shuttle in a zero gravity environment, a globe of the Earth (a thin spherical shell with mass M and radius R) is at rest and centered at the origin. An astronaut throws a tennis ball (mass m and speed V) towards the globe from a position described by $\alpha R\hat{x}$. The ball strikes the globe at a position described by the vector $\vec{r}_C = \frac{1}{\sqrt{3}}R[\hat{x} + \hat{y} + \hat{z}]$. After the collision, the ball is observed to move with a velocity given by

$$\vec{v}_f = \left[\frac{V(\hat{x}+\hat{z})}{\sqrt{(1-\alpha)^2+2}}\right]$$

[Hint: treat the space shuttle as an inertial frame and the tennis ball as a point particle.]

a) Find the displacement vector and the unit vector describing this direction

b) Find the velocity of the globe at the instant after the collision

c) Find the angular momentum of the globe at the instant after the collision

d) Find the angular velocity of the globe at the instant after the collision.

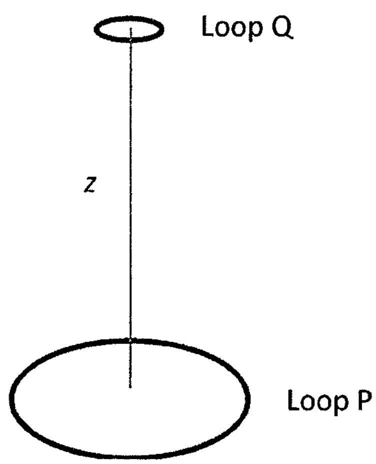
A particle of mass m is constrained to one-dimensional motion within the range $-L < x < L$.

a) In the first part, approximate the potential by an infinite square well.
   1. State the boundary conditions for this case and find the set of eigenfunctions of the system.
   2. Show that the eigenfunctions are orthogonal.

b) In a more realistic scenario the well potential is much larger than the energy of the particle but finite, i.e. $E < V_0 < \infty$. State the boundary conditions for this case and find the (transcendental) relation to determine the solutions.

A small loop of wire (loop Q) with radius a lies a distance z above the center of a large loop (loop P) with radius b as shown in the figure.

a) Consider that the loops lie in planes parallel to each other and z-axis passes through the center of each loop. Let $z > b$. Suppose current I flows in the loop P. Find the magnetic flux through the loop Q.
b) If current I flows in the loop Q. Find the magnetic flux through loop P.
c) Find the mutual inductance between the two loops.
d) Now consider that the two loops are tiny with areas A_1 and A_2 and are separated by a position vector $\vec{r}$. The normal vector of each loop is at arbitrary direction. Find their mutual inductance.



Consider a system of N non-interacting particles. Each particle is fixed in position and can sit in one of two possible states which, for convenience, we will call "spin up" and "spin down". We take the energy of these states to be

$$E_\downarrow = 0 \qquad \text{and} \qquad E_\uparrow = \varepsilon$$

a) Assuming that $N_\uparrow$ is the number of particles in the spin up state and $N_\downarrow$ is the number of particles in the spin down state, what is the number of states in this system?

b) What is the entropy of the system?

c) Use Stirling's formula for large N: $\log N! \approx N \log N - N$, derive the thermal equation of state relating the temperature with the entropy as a function of total energy $E = N_\uparrow \varepsilon$

d) Using the same approximation, show that the entropy maximizes at $N_\uparrow / N \to 1/2$

e) Interpret the temperature at either side of the maximum

Consider the Lagrangian of a two-body problem with the potential that depends on the relative coordinate $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$:

$$L = \frac{1}{2}\left(m_1\left|\dot{\vec{r}}_1\right|^2 + m_2\left|\dot{\vec{r}}_2\right|^2\right) - U(\vec{r})$$

a) What is the number of degrees of freedom in this problem?

b) Rewrite the Lagrangian using the center of mass coordinate and the relative coordinate

c) Using the results from part (b) derive the Lagrangian for the central force case (U depends only on $r \equiv |\vec{r}_1 - \vec{r}_2|$) in the two-dimensional polar coordinate system.

d) Using the Euler-Lagrange equations of motion, show that the angular momentum in the considered central force problem is conserved.

e) Find the Hamiltonian corresponding to the Lagrangian obtained in part (c)

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Classical Mechanics 600-2

A ball of radius a and of mass M is homogeneously filled with density ρ_0. The ball rotates about a fixed axis attached and tangent to the sphere at a point P.

a) Write the mass density $\rho(\vec{r},t)$ with respect to a body-fixed system centered in the center of mass and express ρ_0 in terms of M.

b) What is the same density function $\rho(\vec{r},t)$ as seen from the space-fixed system centered in P and the simplest choice of the coordinate axes?

c) Give the inertia tensor in the body-fixed system of (a).

d) What is the moment of inertia for rotation about a tangent to the ball in P?

e) Find the equations of motion for a rotation about a tangent to the ball in P in body-fixed and spaced-fixed coordinates.

The spin-orbit interaction Hamiltonian in atoms with many electrons is given as

$$\hat{H} = \frac{1}{2m_e c^2 r}\left(\frac{\partial V}{\partial r}\right)\hat{L}.\hat{S} = k\hat{L}.\hat{S}$$

Where V is the Coulomb potential of an electron in the field of the atom, $\hat{L}$ is the angular momentum operator, $\hat{S}$ is the spin operator. Consider k as a coupling factor.

(a) Express $\hat{H}$ in terms of $\hat{J}$, $\hat{L}$ and $\hat{S}$, where $\hat{J}$ is the total angular momentum operator.

Now, consider an electron in the p state in an atom (*i.e.*, $l=1$).

(b) Find the eigenvalues of $\hat{H}$.

(c) Find the degeneracies of each eigenvalue.

Consider a hydrogen atom in the ground state in the presence of a uniform, time-independent, static external electric field $\vec{E}(x) = E_0\hat{z}$. Spin and relativistic effects are to be neglected.

a) Write the Hamiltonian for the hydrogen atom in the presence of the electric field in terms of the center of mass and relative coordinates. Define all symbols used.

b) Find an expression for the lowest order non-vanishing shift ΔE in the ground state energy using time-independent perturbation theory. Indicate clearly which states $|nlm\rangle$ occur in the relevant non-vanishing matrix elements, i.e., which values of *n, l, m* contribute. The matrix elements do not need to be evaluated.

c) Evaluate the expectation value of the dipole moment $\langle\vec{d}\rangle$ to lowest non-vanishing order using the perturbed ground state wave function.

d) The polarizability α of the atom can be defined in terms of the energy shift by (i) $\Delta E = -\frac{1}{2}\alpha\varepsilon_0^2$, or in terms of the induced dipole moment $\vec{d}$ by (ii) $\langle\vec{d}\rangle = \alpha\vec{\varepsilon}$. Make an order of magnitude estimate of the polarizability α. Give your answer in Bohr units.

e) At roughly what value of ε_0 does perturbation theory fail to be a good approximation?

Consider a spin-1/2 particle in a static magnetic field $\vec{B}_0$ along the z-direction such that the unperturbed Hamiltonian for the system is $H_0 = -\gamma B_0 S_z$. For times $t \leq 0$, the system is in the state $|+\rangle$ with eigenvalue $S_z = +\frac{1}{2}\hbar$, using the notation $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in the standard basis formed by the eigenvectors of S_z.

At time $t = 0$, a perturbation is turned on in the form of an additional magnetic field given by

$$\vec{B}_1(t) = \hat{i} B_1 \cos\omega t + \hat{j} B_1 \sin\omega t$$

where $\hat{i}$ and $\hat{j}$ are unit vectors along the x- and y-directions, respectively.

Let $\omega_0 = \gamma B_0$, $\omega_1 = \gamma B_1$, and $\omega_2^2 = (\omega_0 + \omega)^2 + \omega_1^2$.

(a) Write out the perturbation Hamiltonian $H_1(t)$.

(b) Now write $H_1(t)$ in matrix form in the $|+\rangle, |-\rangle$ basis.

(c) Using first-order time-dependent perturbation theory, find the probability that the system will be found with $S_z = -\frac{1}{2}\hbar$ at the end of the time interval $0 \leq t \leq T$.

(d) The exact result for the probability in part (c) is

$$\left(\frac{\omega_1}{\omega_2}\right)^2 \sin^2(\omega_2 T/2)$$

  (i) Show that in the appropriate limit of T, this result agrees with your result from part (c).

  (ii) Do the same for the appropriate limit of B_1.

  (iii) Consider separately and carefully the case at the resonance condition $\omega = -\omega_0$.

In spherical coordinates, the standard series expression to the Laplace equation in terms of the spherical harmonics is

$$\Phi(\vec{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_{l,m} r^{l} + B_{l,m} r^{-(l+1)} \right] Y_{l,m}(\theta,\phi),$$

For real solutions one has the constraints

$$A_{l,m} = (-1)^m A^*_{l,-m}, \quad B_{l,m} = (-1)^m B^*_{l,-m}, \text{ resulting from } Y_{l,m}(\theta,\phi) = (-1)^m Y^*_{l,m}(\theta,\phi)$$

Consider the electrostatic potential, Φ for the region $y > 0$. The region is bounded by a grounded perfectly conducting plane at y=0. A charge distribution, $\rho_{true}(\vec{x})$, is present in the region $y > 0$. The charge distribution is localized.

a) Show that this boundary condition due to the conducting plane requires that the real part of all the $A_{l,m}$ and $B_{l,m}$ coefficients must be zero. Similarly, show that the boundary condition requires all $A_{l,0}$ and $B_{l,0}$ coefficients to vanish.

b) Now consider the multi-pole expansion

$$\Phi(\vec{x}) = \frac{1}{\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} r^{-(l+1)} Y_{l,m}(\Omega) q_{l,m} \text{ , where}$$

$$q_{l,m} = \int_{all\ space} d^3x' Y^*_{l,m}(\Omega') r'^{l} \rho(\vec{x}')$$

Express $\rho_{tot}(\vec{x}')$ in terms of $\rho_{true}(\vec{x}')$ in the allowed $y > 0$ region and find an appropriate $\rho_{image}(\vec{x}')$ in the forbidden $y < 0$ region, consistent with the boundary condition of the grounded *x-z* plane.

c) Consider in particular $\rho_{true}(\vec{x}') = Q\,\delta(x')\,\delta(z')\,\delta(y' - \frac{d}{2})$. What is the exact potential $\Phi(\vec{x})$ in the allowed $y > 0$ region, and what is the first non-vanishing multi-pole contribution for $r \to \infty$?

Consider a wave guide of rectangular cross section along the z-axis. The surfaces of the wave guide (positioned at x=0,a and y=0,b and let a>b) are assumed to be ideally conducting.

a) State the boundary conditions for the $\vec{E}$ and $\vec{B}$ fields.

b) Consider a wave propagating in the z direction

$$\vec{E}(x,y,z,t) = \vec{E}(x,y)\,e^{i(kz-\omega t)} \quad \text{and} \quad \vec{B}(x,y,z,t) = \vec{B}(x,y)\,e^{i(kz-\omega t)}$$

Determine the amplitudes $\vec{E}(x,y)$ and $\vec{B}(x,y)$ satisfying the boundary conditions and obtain a relation for the wave frequency.

c) Let $E_z = 0$ (TE mode). Find the frequency condition for this case and determine all electric field components.

d) For the lowest frequency of the TE mode show that the angle α between the normal to $\vec{E}$ and the x axis direction is given by

$$\cos\alpha = \mp\frac{\pi}{a}\frac{1}{\sqrt{\left(\frac{\pi}{a}\right)^2 + k^2}}$$

and that the phase velocity of the wave is given by $v_{ph} = \dfrac{c}{\sin\alpha}$.

e) Does this type of wave guides allow TM modes (B_z=0) or TEM modes (E_z=B_z=0)? In case that such modes exist, find the frequency conditions for these modes.

A linear antenna of length d along the z-axis is center-fed at z=0 with a current density $\vec{J}(x')e^{-i\omega t} = I_0 \cos(kz')\,\delta(x')\,\delta(y')\,\hat{e}_z\, e^{-i\omega t}$. At first, we may leave $k \equiv \frac{2\pi}{\lambda} \equiv \frac{\omega}{c}$ unspecified, and define $\vec{k} \equiv k\,\hat{n}$, where $\hat{n}$ is the unit vector in the direction of the observation point $\vec{x} = r\,\hat{n}$. In the radiation zone, where $r >> d$ and $r >> \lambda$, we have $\vec{A}(\vec{x}) \approx \frac{e^{ikr}}{cr}\int d^3\vec{x}'\,\vec{J}(\vec{x}')\,e^{-ik\hat{n}\cdot\vec{x}'}$

a) Perform the integration exactly, and so determine $\vec{A}(\vec{x})e^{-i\omega t}$.

b) Now, the exact $\vec{A}(\vec{x})$ that you found has the form $\vec{A}(\vec{x}) \approx I_0 \frac{e^{ikr}}{ckr} f(\theta)\hat{e}_z$ in the radiation zone. Determine the radiated power per solid angle $dP/d\Omega$ for that form of $\vec{A}(\vec{x})$

Now, current cannot flow out of the edge points of the antenna at $z = \pm\frac{d}{2}$, or that would violate the continuity equation (and "integration by parts" procedures based on that). So, we must require $k\frac{d}{2} = \frac{\pi}{2} + n\pi$, where n=0, 1, 2, ...

c) Assume that $kd = \pi$ from now on, and show that $f(\theta) = \frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$ in that case. Write also the corresponding $dP/d\Omega$.

d) Now, although the dipole approximation is not theoretically justified for $kd = \pi$, let us make it anyway. Find the dipole moment $\vec{p}$ of the antenna, and $\vec{A}^{(d)}(\vec{x})$ in the dipole approximation. Then determine the corresponding radiated power $dP^{(d)}/d\Omega$ and compare it with the exact $dP/d\Omega$

Consider a gas of N identical bosons with spin 0 and mass M confined in an external isotropic three-dimensional harmonic oscillator potential. The energy levels of each boson in this potential are $E_n = n\hbar\omega$ and the degeneracy of the level with n quanta is $(n+1)(n+2)/2$.

(a) Find a continuous approximation for the density of states $g(E)$ for a boson confined by this external potential, assuming that $n >> 1$.

(b) Find the expression of the Bose-Einstein condensation temperature T_c of the system in terms of the oscillation frequency ω and the number N of bosons.

(c) For $T < T_c$, find the total energy U of the system, and find the average energy per particle $<E>$ in terms of kT.

(d) The external potential effectively confines each particles inside a volume V, roughly equal to the cube of the particle oscillation amplitude. The oscillation amplitude, in turn, can be estimated by setting the particle's total energy (of order kT) equal to the maximal potential energy of the oscillator. By making these associations, derive an expression for the condensation temperature T_c in terms of the number density N/V.

A molecule has two normal modes of vibrations, with natural frequencies ω_1 and $\omega_2 = 2\omega_1$. Thus, the energy spectrum of the molecule comprises two subsets of levels, with energies $E_j^{(1)} = j\hbar\omega_1$, j=0,1,2,..., and $E_k^{(2)} = k\hbar\omega_2$, k=0,1,2, ... Assume that the molecule is in contact with a reservoir at temperature T. You may set $x = \exp[-\beta\hbar\omega_1]$ for convenience.

a) Determine the canonical partition function $Q_1(\beta)$ for the molecule

b) What is the probability P that the molecule has an energy less than $\frac{5}{2}\hbar\omega_1$.

[Hint: it may help you to sketch a diagram of the molecule energy spectrum]

c) Verify that $P < 1$ for any $0 < x < 1$

d) What is the probability P in the high and low temperature limits?

Now suppose that we have a system of N such molecules in a gas or loosely bound in a molecular solid.

e) What are the contributions of their two vibrational modes to the total energy and to the heat capacity of those systems?

f) What are those contributions in the high and low temperature limits?