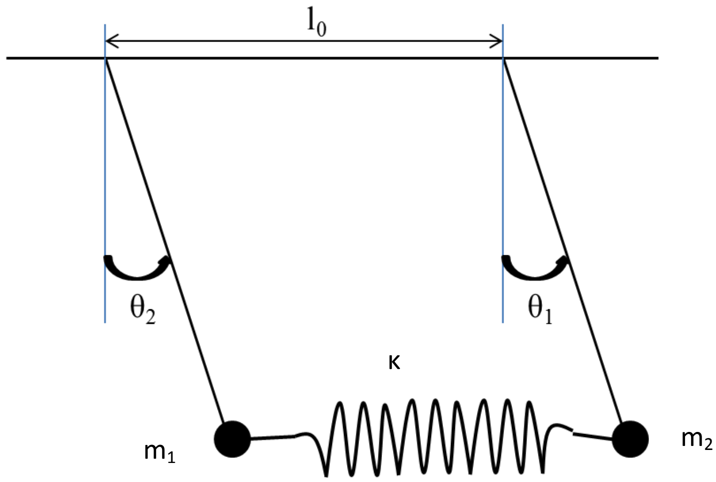
Two coupled pendula, each of length *l* and mass *m*, are supported from a ceiling at a horizontal separation $l_0$. The masses are connected by a spring with spring constant κ and of relaxed length equal to $l_0$, as shown in the Figure.




a) Construct a lagrangian for this system of two coupled pendula in terms of appropriate generalized coordinates and velocities.

b) Assume small oscillations about the equilibrium. Expand the trigonometric functions in the langrangian, retaining only terms linear or quadratic in $\theta_1$ or $\theta_2$

c) From the ensuing equations of motion, construct equations for $\theta_1+\theta_2$ (which is proportional to the displacement of the center of mass from its equilibrium location), and $\theta_1-\theta_2$ (which describes the relative separation). What are the corresponding angular frequencies?

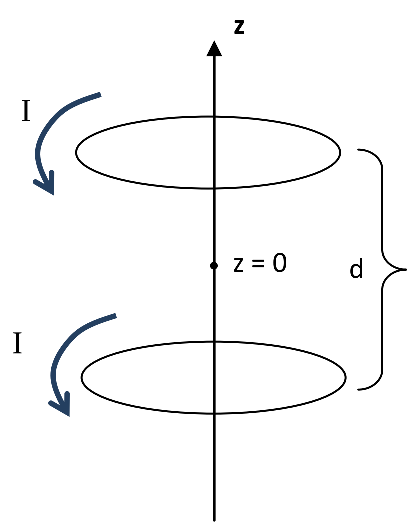
d) What is the general solution for $\theta_1$ as a function of time? What frequencies appear? Under which circumstances would only a single frequency appear, and what would the corresponding motion for $m_1$ (and $m_2$) be? (The modes of motion that occur with a single frequency are called the normal modes of the system).

The magnetic field on the axis of a circular current loop is far from uniform. It falls of sharply with increasing z (assuming z-axis passes through the axis of the loop). We can produce nearly uniform magnetic field by using two current loops at distance d apart.

(a) Consider that the radius of the loop is $R$ and current $I$ is flowing in the counter-clockwise direction. Calculate the magnetic field as a function of $z$ when there is only one loop.

(b) Now, consider two current loops with radius $R$, current $I$ flowing in the counter-clockwise direction in each loops. They are separated by a distance $d$ as shown in the figure below. Find the magnetic field as a function of $z$, and show that $\partial B/dz$ is zero at the point midway between them ($z = 0$).

(c) For a certain value of d, the second derivative will also vanish at $z = 0$. Determine $d$ such that $\partial^2 B/dz^2=0$ at the mid point and find the resulting magnetic field.



Suppose an electron is in the spin state $\chi = A\begin{pmatrix} 1-2i \\ 2 \end{pmatrix}$ .

a) Determine the normalization constant A.

b) When $S_z$ is measured, what are the probabilities for measuring $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$?

c) What is the expectation value of $S_z$?

d) When $S_x$, is measured, what are the possible outcomes and what is the probability of each?

e) What is the expectation value of $S_x$ ?

f) What is the expectation value of $S_y$ ?

g) Suppose $S_z$ was measured and the outcome was $+\frac{\hbar}{2}$ , and then $S_x$ is measured, what are the possible outcomes and what are the probabilities for these outcomes?

The equation of state for a certain non-ideal gas is given by

$$P = \frac{NkT}{V - Nb} - \frac{aN^2}{V^2}$$

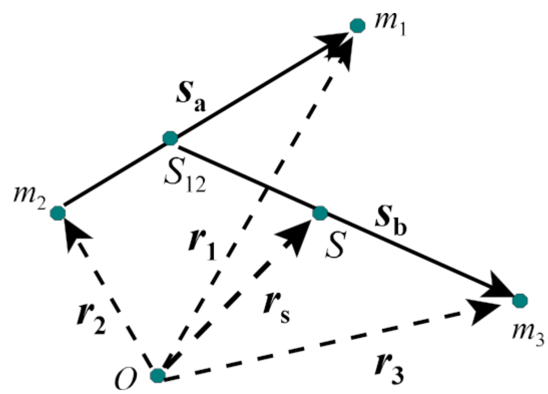
Where $N$ is the number of particles and $a$, $b$, and $k$ are constants.

(a) Using the first and second laws of thermodynamics, find the differential of the internal energy $dE$ (Use the notation $E = U$ as the internal energy) in terms of $dT$ and $dV$. Express the answer in terms of the given quantities, plus $V$, $T$, and the heat capacity $C_V$.

(b) Show that $C_V$ is independent of the volume, that is $\left(\frac{\partial C_V}{\partial V}\right)_T = 0$.

(c) Assuming $C_V$ is independent of temperature, find an expression for the entropy S(T,V) of this gas.

In a system of three mass points $m_1$, $m_2$, $m_3$ let $S_{12}$ be the center of mass (c.m.) of mass points 1 and 2 and $S$ the center of mass of the whole system.

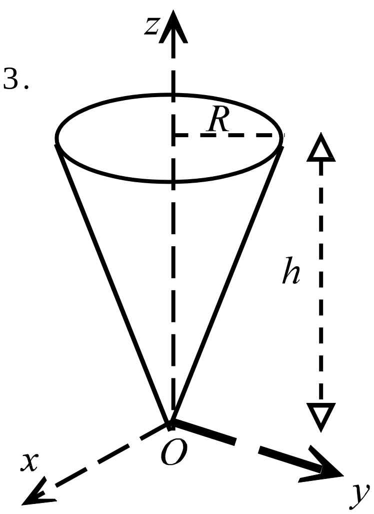



a) Express the coordinate's $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$ in terms of $\boldsymbol{r}_s$, $\boldsymbol{s}_a$, and $\boldsymbol{s}_b$ as defined in the figure.

b) Calculate the total kinetic energy in terms of the coordinates $\boldsymbol{r}_s$, $\boldsymbol{s}_a$, $\boldsymbol{s}_b$ and interpret the results.

c) Write the total angular momentum in terms of the new coordinates and show that $\sum_i \boldsymbol{l}_i = \boldsymbol{l}_s + \boldsymbol{l}_a + \boldsymbol{l}_b$ , where $l_s$ is the angular momentum of the c.m. and $l_a$ and $l_b$ are relative angular momenta with respect to subsystems. Show that $l_s$ depends on the choice of the origin, while $l_a$ and $l_b$ do not.

Consider a conical top with homogeneous mass density $\rho_0$ as shown in the figure. The volume of a conical top is $V = \pi R^2 h / 3$.




a) Calculate the inertia tensors with respect to its center of mass and with respect to its vertex O.

b) Construct the Lagrangian for force-free motion of the top spinning about its vertex using body-fixed coordinates x′, y′, and z′. They coincide with the drawn space-fixed coordinate system for a static top.

c) In order to describe the motion of the top in space-fixed coordinates, consider the following sequence of rotations from the (unprimed) space-fixed system to the (primed) body-fixed system: $R(\phi,\theta,\psi) = R_{z'}(\psi) R_{\xi}(\theta) R_z(\psi)$, where ξ is the intermediate x-axis (line of nodes). Let $\omega_{i'}$ describe the components of the angular velocity in the body-fixed system, then $\phi(t), \theta(t), \psi(t)$ follow the differential equations:

$$\dot{\phi} = [\omega_{x'} \sin\psi + \omega_{y'} \cos\psi] / \sin\theta$$

$$\dot{\theta} = \omega_{x'} \cos\psi - \omega_{y'} \sin\psi$$

$$\dot{\psi} = \omega_{z'} - [\omega_{x'} \sin\psi + \omega_{y'} \cos\psi] \cot\theta$$

Use these relations to find the angular momentum vector in the body-fixed and in the space-fixed system.

d) Transform the Lagrangian using space-fixed coordinates and determine the equations of motion (in space-fixed coordinates).

e) Are there constants of motion? And if so, what is their physical interpretation? (or: And if so, relate these constants of motion to components of the angular momentum)

The permittivity of a medium filling the space between the electrodes of a spherical capacitor whose radii are a and c is given by $\varepsilon(r) = \varepsilon_1$ for $a \le r < b$ and $\varepsilon(r) = \varepsilon_2$ for $b \le r < c$, where $a < b < c$ .

a) Find the capacitance of this capacitor.

b) Determine the distribution of bound charges ($\sigma_b$ *and* $\rho_b$) in both dielectrics and calculate the total bound charge $Q_b$.

Consider a rectangular waveguide, infinitely long in the *z*-direction with perfectly conducting walls with a width of *2a* (*y*-direction) and a height of *a* (*x*-direction)

a) What are the boundary conditions on the components of the electric and magnetic fields of an electromagnetic wave at the walls of this waveguide?

b) From the Maxwell equations derive the Helmholtz equation for *X(x)* and *Y(y)* for the TM mode that propagates along the *z*-direction, and that has electric field along the *x*-direction: $E_x = X(x)Y(y)e^{i(kz-\omega t)}$, where $k$ is the wave number and ω is the angular frequency. Solve this to find the lowest TE mode in the cavity and its frequency.

c) What are the phase and group velocities for the modes in this waveguide?

An infinitely long cylindrical wire of radius $R$ at rest along the z-axis of a laboratory frame K carries both positive $\rho_+$ and negative $\rho_-$ charge densities, both uniform throughout the wire. The net charge density, however, is neutral, i.e., $\rho = \rho_+ + \rho_- = 0$ in K. On the other hand, $\rho_+$ is flowing with constant velocity $\vec{v}_{0+} = v_{0+}\,\hat{e}_z$, $v_{o+} > 0$ in K, while $\rho_-$ is standing stationary in K, having $\vec{v}_{0-} = 0$. Correspondingly, the wire in K carries a positive current density $\vec{j}_+ = j_+\,\hat{e}_z$, $j_+ = \rho_{0+}\, v_{0+} > 0$, and no negative current density $\vec{j}_- = j_-\,\hat{e}_z = \rho_-\,\vec{v}_{0-} = 0$. So, the wire carries a net current density $\vec{j} = \vec{j}_+ + \vec{j}_- = \vec{j}_+$ in K.

(a) Recalling the relativistic expression $j^\mu = (c\rho, \vec{j})$ of a four-current, determine $j_+^\mu$, $j_-^\mu$, and $j^\mu = j_+^\mu + j_-^\mu$, $\mu = 0,1,2,3$ in K. Also determine their corresponding Lorentz invariants $j^\mu j_\mu = -c^2\rho^2 + \left|\vec{j}\right|^2$.

Now consider a sliding frame K$'$ that moves with a constant velocity $\vec{v}$ along the z-axis of K. Event-coordinates $x^{\mu'} = (ct', \vec{x}')$ in K$'$ are related to those in K by a Lorentz transformation with $t' = \gamma\left(t - \frac{v}{c^2} z\right)$ and $z' = \gamma(z - vt)$, $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$.

(b) Determine the corresponding Lorentz transformation between K and K$'$ for all the components of the three four-currents $j_+^{\mu'}$, $j_-^{\mu'}$, $j^{\mu'}$.

Now assume in particular that $\vec{v} = \vec{v}_{0+}$. Then in K$'$ $\rho_+'$ is stationary with a corresponding $\vec{j}_+' = 0$, while $\rho_-'$ is flowing backward with constant velocity ($-\vec{v}$), yielding $\vec{j}_-' = -\rho_-'\,\vec{v}$.

(c) Examine the symmetry, or lack of it, between both charge and current densities in K and K$'$. Explain $\rho_+ = \gamma\rho_+'$ and $\rho_-' = \gamma\rho_-$ in terms of Lorentz dilations. Draw a sketch of the wire and the corresponding charge and current densities as seen in the K and K$'$ frames, moving relatively to each other, in order to help visualization of that explanation. Why then is the wire negatively charged in K$'$, whereas it is neutral in K?

(d) Using Gauss and Ampère laws, determine the electric and magnetic fields outside the wire as they appear in both frames K and K$^{/}$. Interpret the results and verify that $\vec{E}.\vec{B}=\vec{E}^{/}.\vec{B}^{/}$ *and* $\left|\vec{B}\right|^2-\left|\vec{E}\right|^2=\left|\vec{B}^{/}\right|^2-\left|\vec{E}^{/}\right|^2$ are Lorentz invariants, using Gaussian units.

Consider the case of a simple harmonic oscillator. The stationary states of a simple harmonic oscillator can be written in terms of the raising operator as

$$\left|n\right\rangle = \frac{1}{\sqrt{n!}}(a_{+})^{n}\left|0\right\rangle$$

Among the stationary states only n=0 hits the uncertainty limit $(\Delta x)(\Delta p) = \hbar/2$ while in general $(\Delta x)(\Delta p) = (2n+1)\hbar/2$. But certain linear combinations (known as **coherent states**) minimize the uncertainty product. They are eigenstates of the lowering operator $a_{-}$: $a_{-}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle$ where the eigenvalue $\alpha$ can be any complex number.

(a) calculate $<x>$, $<x^2>$, $<p>$, $<p^2>$ in the state $|\alpha\rangle$.

(b) Find $\Delta x, \Delta p$ and show that $(\Delta x)(\Delta p) = \hbar/2$ .

(c) Coherent states can be expanded in terms of energy eigenstates as

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n \, |n\rangle$$

Show that the expansion coefficients are $c_n = \frac{\alpha^n}{\sqrt{n!}} c_0$

(d) Determine $c_0$ by normalizing $|\alpha>$.

(e) Is the ground state $|n=0>$ itself a coherent state? If so, what is its eigenvalue $\alpha$ of $a_{-}$?

**Useful results:**

$$\left\langle x\middle|0\right\rangle = \Psi_0(x) = \left[\frac{m\omega}{\pi\hbar}\right]^{\frac{1}{4}} e^{-\left(\frac{m\omega}{2\hbar}\right)x^2}$$

$$a_{\pm} = \sqrt{\frac{m\omega}{2\hbar}}\left(x \mp i\frac{p}{m\omega}\right)$$

a) Use the variational principle with the trial wave function

$$\psi(x) = \left(\frac{2\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2}$$

to prove that, in one dimension, a potential that is everywhere attractive always has at least one bound state. Explain clearly why demonstration with this convenient but arbitrary trial function constitutes a proof.

b) Use the ground state for the three dimensional harmonic oscillator

$$\psi(\vec{r}) = \left(\frac{2\alpha}{\pi}\right)^{3/4} e^{-\alpha r^2}$$

as a trial wave function to calculate the approximate ground state energy of the hydrogen atom.

c) Compare your result in part (b) to the exact value

$$E_0 = -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2$$

and comment on the difference.

An ammonia molecule consists of three hydrogen atoms in a plane plus a nitrogen atom *above* or *below* the plane so that the whole molecule looks like a pyramid. Assuming all other degrees of freedom are frozen, the two symmetrical positions of the nitrogen yield two quantum mechanical states of the ammonia, which we denote as $|1\rangle$ and $|2\rangle$.

a) $|1\rangle$ and $|2\rangle$ are not the eigenstates of the ammonia because there is a finite probability for the nitrogen to tunnel from one position to another. The Hamiltonian matrix for the ammonia in the above basis is thus nondiagonal

$$\begin{pmatrix} E_0 & A \\ A & E_0 \end{pmatrix},$$

where $A > 0$ is proportional to the tunneling probability. Solve for the energy eigenvalues $E_\pm$ and eigenstates $|\pm\rangle$ of the ammonia, where $E_+ > E_-$. Experimentally, the difference $E_+ - E_-$ is about $10^{-4}$ eV, and thus at room temperature ($k_B T$=1/40 eV) the two states of ammonia are populated with almost equal probabilities.

b) To separate out the ammonia molecules at different energies one can pass a beam of ammonia through an electric field ε, in which the Hamiltonian matrix of the ammonia becomes

$$\begin{pmatrix} E_0 + \mu\varepsilon & A \\ A & E_0 - \mu\varepsilon \end{pmatrix}$$

where μ is a constant. Find the energy eigenvalues $E_\pm(\varepsilon)$ and expand them in the limit of small ε. Consider a beam of ammonia travelling in the $\hat{z}$ direction. Show that the two states can be separated by a spatially varying electric field $\varepsilon(x)\,\hat{x}$.

c) After exiting the above field, the beam of ammonia is entirely in the high-energy state $E_+$. Now let the beam pass through a cavity in which there is a (different) oscillating electric field.

$$\varepsilon(t) = 2\varepsilon_0 \cos(\omega_0 t)$$

where $\omega_0$ is set to the resonance condition $\omega_0 = (E_+ - E_-)/\hbar$. Transform the Hamiltonian matrix of the ammonia in the cavity to a new basis $|\pm\rangle$. From that find the simplified time-dependent Schroedinger equation for $C_+(t)$ and $C_-(t)$, the coefficients in a general state $|\psi(t)\rangle = C_+(t)e^{-iE_+t/\hbar}|+\rangle + C_-(t)e^{-iE_-t/\hbar}|-\rangle$.

Consider a system of non-interacting fermions.

a) Using the grand canonical partition function, derive the Fermi-Dirac distribution function.

Now assume that each fermion has non-degenerate and equally spaced energy levels (spacing equal to $\eta$) and that all fermions are forced to be in the same spin orientation state. We wish to analyze this system of non-interacting fermions in the temperature range $\eta << kT << \varepsilon_F$, where $T$ is the absolute temperature and $\varepsilon_F$ is the Fermi energy. Notice that this system of "Fermi oscillators" is not the familiar ideal gas of fermions, which has a different single particle energy spectrum and degeneracy.

b) Write out the density of states and use this together with the Fermi-Dirac distribution to obtain the energy of this system.

[Hint: Integrate by parts and argue that the lower limit of integration can be extended to $-\infty$.]

c) Find the heat capacity of this system.

Now analyze this system in the microcanonical ensemble, as follows:

d) Draw an energy level diagram of the ground state configuration (e.g., using filled and open circles to represent filled and empty states).

e) If we take the energy of the system in the ground state to be zero, the energy of the system can be written as $E = q\eta$, where $q$ is the number of units of energy above the ground state of the system. Draw energy level diagrams for $q$ = 1, $q$ = 2, and $q$ = 3 showing all possible microstates.

f) Argue that the number of microstates for a given $q$ is equal to *P(q)*, where *P(q)* is the number of unrestricted partitions of the integer $q$ (i.e., the number of distinct ways that $q$ can be written as a sum of non-zero integers. For example: $P(1)$ = 1 [1]; $P(2)$ = 2 [2, 1+1]; $P(3)$ = 3 [3, 2+1, 1+1+1]; $P(4)$ = 5 [4, 3+1, 2+2, 2+1+1, 1+1+1+1]; etc.).

g) For large $q$,

$$P(q) \approx \frac{e^{\pi\sqrt{2q/3}}}{4\sqrt{3}q}$$

a result due to Ramanujan and Hardy. Use this result to find the entropy as a function of $q$.

h) Find the temperature as a function of $q$.

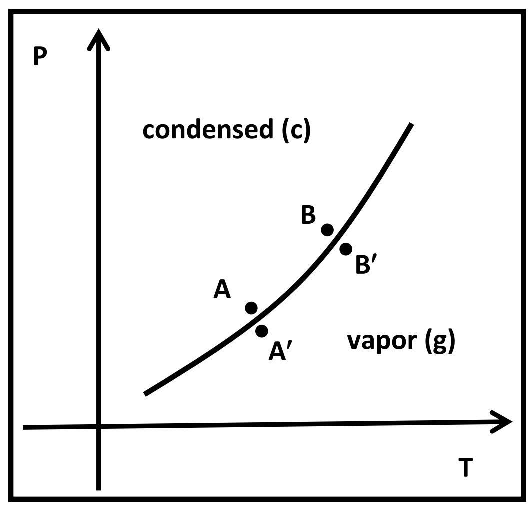
i) Find the energy and hence the heat capacity and compare your result with part (c).

MS/Ph.D. Comprehensive Exam, Fall 2011
Statistical Mechanics 600-2

This problem deals with the equilibrium of a condensed (solid or liquid) phase (c) and a vapor phase (g) of a single-component system. Let us first adopt the thermodynamic view point and establish the Clapeyron Equation $\frac{dP}{dT}=\frac{s_g - s_c}{v_g - v_c} = \frac{l(T)}{T(v_g - v_c)}$ for the vapor pressure P = P(T) between the two phases (c) and (g).

(a) Explain the meaning and symbols of all the quantities in the Clapeyron equation and derive it or justify it as you wish or may. One possibility is to consider the two-phase coexistence line in a P-T diagram and apply the Gibbs-Duhem relation $\Delta\mu = -s\Delta T + v\Delta p$ for the equilibrium condition $\mu_g(T,P) = \mu_c(T,P)$. See Figure.

(b) Now suppose that the vapor phase (g) is modeled by an ideal gas with an equation of state $v_g = \frac{kT}{P}$, where $k$ is the Boltzmann constant (if you consider specific quantities per molecule, or the gas constant if you consider specific quantities per mole). Then suppose that the condensed phase (c) has a density $\frac{1}{v_c} >> \frac{1}{v_g}$. Thus solve the Clapeyron equation for the vapor pressure P = P(T), further assuming that the latent heat of vaporization $l = T(s_g - s_c)$ is positive and essentially independent of temperature T in this model. If water boils at $T_0$ = 373 K and at atmospheric pressure $P_0$ = 1 atm at sea level, does it boil at a higher or a lower temperature T on a high mountain, where the pressure P is lower than $P_0$?




Now we wish to formulate the same two-phase equilibrium problem from a statistical mechanics view point, using the Grand Canonical ensemble.

(c) Model the vapor phase (g) again as a classical ideal gas of $N_g$ indistinguishable particles.

Derive the equation of state $z_g = \dfrac{\overline{N}_g}{Q_{1g}}$ for the ideal gas fugacity. Explain the meaning of all your symbols and relations.

(d) Model the condensed phase (c) as a system of $N_c$ distinguishable non-interacting Planck oscillators of energies $\varepsilon_m = -\varepsilon_0 + m\hbar\omega,\ m = 0,1,2,3,...$ all starting from a binding energy $(-\varepsilon_0) < 0$ that keeps the condensed phase bound relatively to the zero of energy of the free particle gas phase. Derive the equation of state $z_c \cong \dfrac{1}{Q_{1c}^{(o)}} e^{\beta\varepsilon_0}$ for the condensed phase fugacity.

(e) From the basic condition of phase equilibrium establish the statistical mechanics equivalent of the Clapeyron equation proposed in (b) and compare the corresponding solutions. We should clearly assume that $\hbar\omega << \varepsilon_0$ (why?)