Ph.D. Comprehensive Examination

Physics Department

Fall 2009

Thursday, October 22, and Friday, October 22, 2009

Room 133 - Hannan Hall

GENERAL INSTRUCTIONS:

This examination is divided into four sections as follows:

Thursday, March 26, 2009

9:00 a.m. - 12:00 Noon	Classical Mechanics - **2 questions**
1:00 P.M. - 5:00 P.M.	E & M - **3 questions**

Friday, March 27, 2009

9:00 a.m. - 12:00 Noon	Stat Mech. - **2 questions**
1:00 P.M. - 5:00 P.M.	Quantum Mech. - **3 questions**

In each of the four subject areas, you may answer **the 500-level question** in place of a 600-level question.

DO EACH PROBLEM IN A SEPARATE BLUE BOOK

PUT YOUR NAME ON EACH BOOK

LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS
FOR EXAMPLE: Mechanics 600-1

MS Comprehensive Examination

Physics Department

Fall 2009

Thursday, October 22 and Friday, October 23, 2009

Room 133 - Hannan Hall

GENERAL INSTRUCTIONS:

This examination is divided into four sections as follows:

Thursday, March 26, 2009	
9:00 a.m. - 12:00 Noon	Classical Mechanics - 2 questions
1:00 P.M. - 5:00 P.M.	E & M - 2 questions
Friday, March 27, 2009	
9:00 a.m. - 12:00 Noon	Thermodynamics/Stat. Mech. - 2 questions
1:00 P.M. - 5:00 P.M.	Modern Physics/Quantum Mech. - 2 questions

DO EACH PROBLEM IN A SEPARATE BLUE BOOK

PUT YOUR NAME ON EACH BOOK

LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS
FOR EXAMPLE: Mechanics 600-1 or Electromagnetism 500-2

YOU MUST DO TWO (AND ONLY TWO) OF THE QUESTIONS IN EACH AREA.

OF THE TOTAL OF EIGHT QUESTIONS THAT ARE DONE, A MINIMUM OF TWO MUST BE FROM THE 600-LEVEL QUESTIONS.

CUA

THE CATHOLIC UNIVERSITY OF AMERICA

DEPARTMENT OF PHYSICS

200 Hannan Hall
Washington, D.C. 20064
202-319-5315 • Fax: 202-319-4448
http://physics.cua.edu

RULES FOR THE PHYSICS COMPREHENSIVE EXAMINATION

Fall 2009 (October 22 & 23, 2009)

In order to assure an equitable and fair Comprehensive Examination for all students the following rules will apply to the examination administered by the Department of Physics.

1. The examination will be **closed book**. Students will not be permitted to bring to the examination room materials such as notes, reference books, calculators, or other aids of any form. The Physics Department will supply copies of the *CRC Mathematical Handbook*, Schaum's *Mathematical Handbook*, the *Table of Functions* by Jahnke and Emde, and the *NBS Handbook of Mathematical Functions*, for use during the examination.

   Students who want to use their own mathematical tables and/or handbooks must get permission from the Chairman prior to the beginning of the comprehensive examination. These tables and/or handbooks will remain under the control of the department until after the examination is completed, but will be available to their owners during the examination period.

   The Physics Department will supply calculators for use during the examination.

2. If needed, students will be allowed a short break, not to exceed ten minutes, during the examination period. While absent from the examination room students may not consult material or other sources of information concerning any matter pertaining to the examination. Students who leave the examination room will turn in all of their examination material to the proctor before leaving the room. Examination papers will be returned to the students when they reenter the examination room. Students who fail to return within ten minutes will be considered to have completed that portion of the examination.

   Only one student will be permitted to leave the examination room at a time.

a) Show that the moment of inertia of a thin rod of length l about its center of mass is $ml^2/12$.

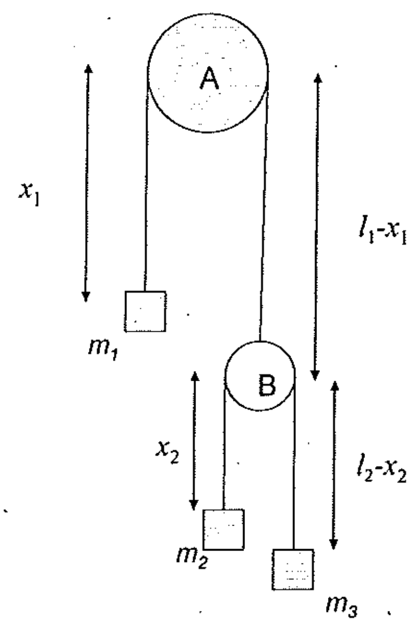
b) A long thin tube of negligible mass is pivoted so that it may rotate without friction in a horizontal plane. A thin rod of mass M and length l slides without friction in the tube. Choose a suitable set of coordinates and write Lagrange's equations for this system.

c) Initially the rod is centered over the pivot and the tube is rotating with angular velocity ω_0. Show that the rod is unstable in this position.

MS Comprehensive Examination
Fall 2009
Classical Mechanics: CM500-2

Below is an example of a double Atwood's machine. The top pulley (A) is fixed; while the bottom pulley (B) is free to move. The masses of both the pulleys are negligible compared to the masses, m_1, m_2, and m_3. The mass m_1 is connected to pulley A by a cord of length l_1 that slides freely around pulley A. The masses m_2 is, likewise, connected to m_3 by a cord of length l_2 that freely slides over the pulley B. The cords, with lengths l_1 and l_2, do not stretch when tension is applied.

a.) Write the Lagrangian for the system in terms of the generalized coordinates.
b.) What are the equations of motion?
c.) What are the expressions for the accelerations $\ddot{x}_1$ and $\ddot{x}_2$?

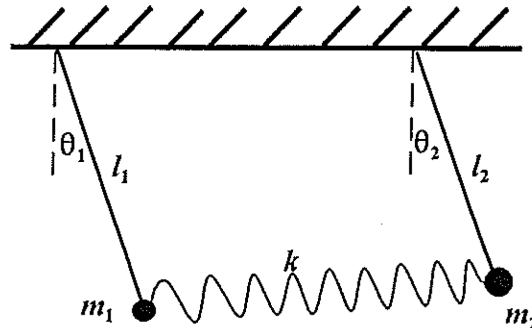


A sphere of mass M and radius R rolls without slipping down a triangular block of mass m that is free to move on a frictionless horizontal surface.

a) Find the Lagrangian and state Lagrange's equations for this system subject to the force of gravity at the surface of the Earth.

b) Find the motion of the system by integrating Lagrange's equation, given that all objects are initially at rest and the sphere's center is at a distance H above the surface.

Consider the system of two pendulums connected by a weightless spring whose length is equal to the distance between the suspension points. The pendulums consist of massless rods of l_1 and l_2 and bobs of masses m_1 and m_2. For the following calculations consider only small angular displacements of the bobs and regard the spring as effectively horizontal.



1. For the case that the pendulums are identical (m_1=m_2, l_1=l_2) find the normal modes of this system.
   Sketch the characteristic oscillations of the system.
   How do the results change if the spring is very weak (k small) or very stiff (k large)?

2. Find the normal frequencies of the system if the pendulums are different ($m_1 \neq m_2$, $l_1 \neq l_2$).
   How do the normal frequencies behave as $k \to 0$ or as $k \to \infty$?

Charges +q & -q lie at the points (x,y.z) = (a,0,a) & (-a,0,a) above a grounded conducting plane at z = 0. Find

a) the total force on charge +q,

b) the work done against the electrostatic forces in assembling this system of charges,

c) the surface charge density at the point (a,0,0).

A non-conducting sphere of radius R contains a non-uniform charge density

$$\rho(r) = Ar^2 \quad (0 \leq r \leq R).$$

a) Find the total charge of the sphere.

b) Find the electric field at any point *outside* the sphere ($r > R$).

c) Find the electric field at any point *inside* the sphere ($0 < r < R$).

d) Find the **potential** of the surface of the sphere ($r = R$), taking $V = 0$ at infinity.

e) Find the **difference of potential** between the center of the sphere and the surface of the sphere. Assuming $A > 0$, is the center at higher or lower potential than the surface?

PhD Comprehensive Exam
Fall 2009
Electricity & Magnetism: EM600-1

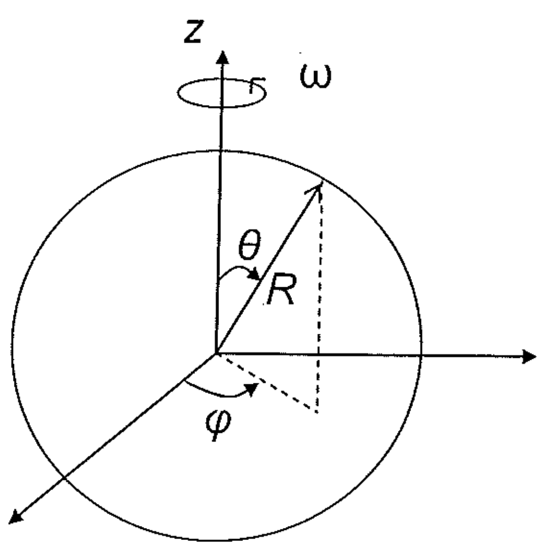
Consider a uniformly charged spherical surface with a total surface charge Q and radius R. It is a vacuum both interior and exterior to this surface. This charge surface rotates at a constant angular velocity ω about the z-axis. It produces a magnetic field that is proportional to ω.

a.) Determine the surface current density $\vec{J}_s$ at the angle θ from the z-axis.

b.) What is the magnetic scalar potential inside and outside of the sphere?

c.) What is the induced magnetic field interior and exterior to the sphere?

d.) Show that the magnetic dipole moment of the spinning sphere is given by:
$$\vec{m} = \frac{QR^2}{3}\vec{\omega}.$$



M.S. / Ph.D. Comprehensive Examination
Fall 2009
Electricity & Magnetism 600-2

A charge e moves at a constant angular velocity ω along a circular orbit or radius a in the xy plane.

1. Calculate the electric dipole moment of the rotating charge.

2. Determine $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ For the far zone ($r \gg 1/k$).

   You may use the approximation for the vector potential: $\vec{\mathbf{A}}(\vec{\mathbf{r}},t) = -i\frac{\mu_0\omega}{4\pi}\vec{\mathbf{p}}(\vec{\mathbf{r}},t)\frac{e^{ikr}}{r}$.

3. Determine the polarization of the radiated electromagnetic wave as seen for observers at different positions, in particular along the z axis and in the xy plane.

4. Calculate the average radiated power per solid angle in an arbitrary direction $\vec{\mathbf{n}}$ and the total radiated power.

5. An oscillating dipole radiates an electromagnetic angular momentum at the rate:

   $$\frac{d\vec{\mathbf{L}}}{dt} = \frac{k^3}{12\pi\varepsilon_0}\mathrm{Im}(\vec{\mathbf{p}}^* \times \vec{\mathbf{p}}).$$

   Calculate the direction and rate of the radiated angular momentum for the rotating charge.

Consider a hollow waveguide with perfectly conducting walls. Inside the waveguide, monochromatic waves propagate in the z direction:

$$\vec{E}(x,y,z,t) = \vec{E}_0(x,y)\,e^{i(kz-\omega t)}, \quad \vec{B}(x,y,z,t) = \vec{B}_0(x,y)\,e^{i(kz-\omega t)} \tag{1}$$

where, in general, $\vec{E}$, $\vec{E}_0$, $\vec{B}$ and $\vec{B}_0$ are *complex* vectors.

In order to fit the boundary conditions in a wave guide, the electric and magnetic fields must have *longitudinal* as well as transverse components:

$$\vec{E}_0 = E_x(x,y)\,\hat{x} + E_y(x,y)\,\hat{y} + E_z(x,y)\,\hat{z}, \quad \vec{B}_0 = B_x(x,y)\,\hat{x} + B_y(x,y)\,\hat{y} + B_z(x,y)\,\hat{z}\,. \tag{2}$$

a) Starting with Maxwell's equations, derive the wave equation satisfied by each component of $\vec{E}$ and $\vec{B}$ for an electromagnetic wave propagating in free space.
NOTE: Treat ω and k as independent variables, i.e. do not assume that $\omega/k = c$.

b) For the fields in the waveguide described by (1) and (2), derive the uncoupled partial differential equations satisfied by E_z and B_z.
Hint: The result for E_z is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right) E_z = 0\,. \tag{3}$$

Note: You may begin with Equation (3) to do the following parts.

c) Consider transverse magnetic (TM) waves ($B_z = 0$, $E_z \neq 0$) of frequency ω in a perfectly conducting rectangular waveguide of height a and width b. Find the solutions for $E_z(x,y)$ by separation of variables, applying the boundary conditions $E^{\parallel} = 0$, $B^{\perp} = 0$ appropriate at the surface of a perfect conductor.

d) For given ω, find the possible wave numbers k which satisfy the above conditions. What is the lowest possible frequency ω for which a wave can propagate in a TM mode in this waveguide?

Consider a classical ideal monatomic gas in thermal equilibrium at temperature T.

(a) Write the expression for the velocity distribution function of the atoms.
(b) Derive the expression for the *most probable speed* of the atoms.
(c) If the gas is atomic hydrogen at a temperature T = 6400 K. estimate the value of the *most probable* speed. The mass of a hydrogen atom is $m = 1.67 \times 10^{-27}$ kg and Boltzmann's constant is $k = 1.38 \times 10^{-23}$ J. K^{-1}.
(d) Calculate the average kinetic energy per atom of this gas. What is the rms speed of an atom in the gas?

M.S. Comprehensive Examination
Fall 2009
Thermodynamics 500-2

An ideal gas is expanded adiabatically from (P_1, V_1) to (P_2, V_2), where P is the pressure. It is then compressed at constant pressure to (P_2, V_1). Finally, the pressure is increased to P_1 at constant volume V_1.

1. Draw the cycle on a P-V plot.

2. Show that the efficiency, η, of the cycle given by: $\eta = 1 - \gamma \frac{V_2/V_1 - 1}{P_1/P_2 - 1}$, where $\gamma = C_P/C_V$.

MS/PhD Comprehensive Examination
Fall, 2009 ~~Spring 2007~~
Statistical Mechanics 600-1

Consider an ideal gas of N non-relativistic fermions (S=½) in a three-dimensional volume V at an absolute temperature T.

A. Derive or provide the expression for $n(\epsilon)d\epsilon$, representing the number of fermions with energy in the range ϵ to $\epsilon+d\epsilon$, in terms of the chemical potential $\mu=\mu(T,N/V)$.

B. Determine $n(\epsilon)$ for $T\to 0$.

C. Determine $\epsilon_F=\mu(T=0,N/V)$.

D. Sketch $n(\epsilon)$ for $kT \ll \epsilon_F$.

E. Determine $n(\epsilon)$ for $\exp(-\mu/kT)=\frac{1}{Z}\gg 1$ in terms of N rather than Z.

F. Sketch $n(\epsilon)$ for $\exp(-\mu/kT)=\frac{1}{Z}\gg 1$.

G. Find the most probable energy ϵ^* for $T\to 0$. Find the corresponding maximum $n(\epsilon^*)$ in terms of N and ϵ_F.

H. Find the most probable energy ϵ^* for $\exp(-\mu/kT)=\frac{1}{Z}\gg 1$. Find the corresponding maximum $n(\epsilon^*)$ in terms of N and kT.

I. Plot and compare on the same graph $n(\epsilon)$ vs. ϵ the two cases: $n(\epsilon)$ for T=0 and $n(\epsilon)$ for $T\gg\frac{\varepsilon_F}{k}$. Which is larger and why?

PhD/Comprehensive Examination
Fall 2009
Statistical Mechanics 600-2

The Hamiltonian of a two-dimensional dipolar molecule of mass m, moment of inertia I, and dipole moment μ, in an electric field E is

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{p_\theta^2}{2I} - \mu E \cos\theta$$

where θ is the angle between $\vec{E}$ and $\vec{\mu}$.

a) Find the canonical partition function of a system of N such classical dipolar molecules confined to an area A.

b) Obtain low-temperature and high-temperature approximations for the specific heat at constant area, C_A, and confirm that the effect of the dipole-field interaction becomes negligible when $\mu E << kT$.

c) The probability of finding a two-dimensional dipolar molecule in a state described by the phase space variables (x, y, p_x, p_y, θ) is proportional to $e^{-\beta H}$. Show that if the electric field is not uniform, then the gas is more dense where the field is higher.

Hints: The modified Bessel function $I_o(u)$ is a monotonically increasing function of $|u|$, with an integral representation:

$$I_o(u) = \frac{1}{\pi}\int_o^\pi e^{u\cos\theta}\,d\theta$$

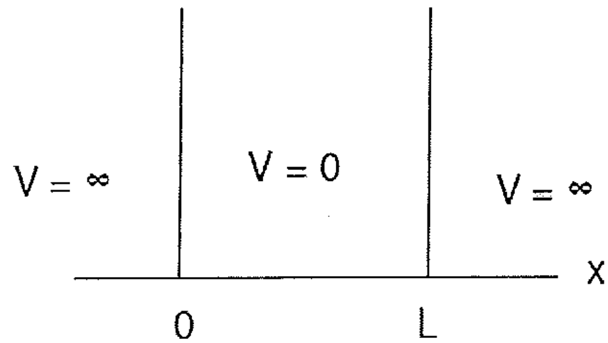
Its power series expansion is:

$$I_o(u) = \sum_{k=o}^{\infty}(u/2)^{2k}/(k!)^2,$$

while for large u:

$$I_o(u) \sim e^u/\sqrt{2\pi u}.$$

A particle of mass m is confined in a 1-d infinite square well: 0 < L< x.



The normalized wavefunctions for the system are

$$\Psi_n(x,t) = \sqrt{2/L}\,\sin\frac{n\pi x}{L}\,\exp\{-iE_n t/\hbar\}\ ,$$

where, E_n is the n[th] energy eigenvalue with n = a positive integer.

(a) Use the relation $\Delta p\,\Delta x \geq \frac{1}{2}\hbar$ to establish that $E = 0$ is not an admissible energy eigenvalue.

(b) Calculate the energy eigenvalues.

(c) Suppose the system consisting of the particle and the well is such that at $t = 0$, its state is described by:

$$\Psi(x,0) = \frac{1}{\sqrt{2}}[\Psi_1(x,0) - \Psi_2(x,0)]\,.$$

Calculate the probability density $\Psi^*\Psi$ for the particle's position for t > 0. Use this to describe the motion of the particle.

(d) Is the motion periodic? If so, what is the frequency?

a) The energy of an electron in an atom is shifted by the so-called spin-orbit interaction, which results in a term of the form $\Delta E = A\langle \vec{L}\cdot\vec{S}\rangle$, where $\vec{L}$ is the orbital angular momentum, $\vec{S}$ is the spin angular momentum, A is a constant , and the brackets $\langle\ \rangle$ denote the expectation value in a particular quantum state. Let the total angular momentum of the electron be $\vec{J} = \vec{L} + \vec{S}$. The magnitudes of the angular momenta $\vec{L}$, $\vec{S}$ and $\vec{J}$ are characterized by the quantum numbers l, s (= ½) and j respectively.

1) Find the expectation value $\langle \vec{L}\cdot\vec{S}\rangle$ in terms of l, s and j. (Hint: Begin by calculating the expectation value of $J^2 \equiv \vec{J}\cdot\vec{J}$.)

2) Consider an electron with orbital angular momentum quantum number $l = 2$.
What are the possible values of j ?
What are the possible values of $\langle \vec{L}\cdot\vec{S}\rangle$?
What are the possible values of the projection quantum numbers m_l , m_s, m_j ?

b) The "2p shell" of an atom consists of all those electrons whose principal quantum number is $n = 2$ and whose orbital angular momentum quantum number is $l = 1$.
State the Pauli exclusion principle in the most general form you can, and use it to explain why the maximum number of electrons in the 2p shell is 6.

Consider a spin-3/2 ($s = 3/2$) particle. The normalized eigenfunctions $|s\,m\rangle$ of the operators S^2 and S_z can be represented by four-dimensional column matrices

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \text{ and } \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

a) Construct matrices representing the operators S^2, S_z, S_+, S_-, S_x, and S_y in this representation.

Hint: The operators $S_\pm$ are defined by the equations

$$S_\pm = S_x \pm i S_y$$
$$S_+|s\,m\rangle = \hbar\sqrt{s(s+1) - m(m+1)}\,|s\;m+1\rangle$$
$$S_-|s\,m\rangle = \hbar\sqrt{s(s+1) - m(m-1)}\,|s\;m-1\rangle$$

b) Of the six matrices constructed in part (a), which are hermitian?

c) An arbitrary spin state in this space can be represented by the column matrix $\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$. Find the expectation value of S_z in this state.

Part (d) does not depend on parts (a)-(c).

d) Consider a particle with spin $s = 3/2$ and orbital angular momentum $l = 2$. What are the possible quantum numbers j, m_j which describe the total angular momentum $\vec{J} = \vec{L} + \vec{S}$?

Consider an isotropic harmonic oscillator in **three** dimensions (k=1,2,3 represent the Cartesian coordinate components):

$$H = \tfrac{1}{2\mu}\vec{\mathbf{p}}^2 + \tfrac{\mu}{2}\omega^2\vec{\mathbf{r}}^2 = \sum_{k=1}^{3} \tfrac{1}{2\mu} p_k^2 + \tfrac{\mu}{2}\omega^2 r_k^2$$

1. Use ladder operators a_k and a_k^+ with $N_k = a_k^+ a_k$ and $[a_j, a_k^+] = \delta_{jk}$ $(j,k = 1,2,3)$ and the simultaneous eigenbasis $|n_1 n_2 n_3\rangle$ to determine the energy levels E_n.

2. Express $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$ in terms of $\vec{\mathbf{a}}$ and $\vec{\mathbf{a}}^+$ with:
$$\vec{\mathbf{a}} = \frac{1}{\sqrt{2\mu\hbar\omega}}(\mu\omega\,\vec{\mathbf{r}} + i\vec{\mathbf{p}})\,;\quad \vec{\mathbf{a}}^+ = \frac{1}{\sqrt{2\mu\hbar\omega}}(\mu\omega\vec{\mathbf{r}} - i\,\vec{\mathbf{p}})$$

3. Evaluate $L_k|n_1 n_2 n_3\rangle$ and calculate the action of L_z and $L_\pm$ on $|n_1 n_2 n_3\rangle$.

4. Express the symmetric rank-2 tensor $A_{ij} = \frac{1}{2}\hbar\omega(a_i^+ a_j + a_i a_j^+)$ in terms or $\vec{\mathbf{r}}$ and $\vec{\mathbf{p}}$. Write H as linear combination of A_{ij}.

5. The quadrupole tensors are defined as
$$Q_0 = \frac{1}{\omega}(2A_{33} - A_{11} - A_{22})\,;\; Q_{\pm 1} = \mp\frac{1}{\omega}(A_{13} \pm iA_{23})\,;\; Q_{\pm 2} = \frac{1}{\omega}(A_{11} - A_{22} \pm 2iA_{12})$$
Express the quadrupole tensors in terms of a_k and a_k^+, and calculate $[L_z, Q_\lambda], [L_\pm, Q_\lambda]$ where λ=0, ±1, ±2.

Hint: $[L_i, A_{jk}] = i\hbar\varepsilon_{ilm}(\delta_{jl}A_{km} + \delta_{kl}A_{jm})$

Let a molecule (approximated by a 3-dimensional free rotator with moment of inertia I and electric dipole moment $\vec{\mathbf{p}}$) be located in a uniform electric field $\vec{\mathbf{E}}$. Since $\vec{\mathbf{p}}$ is not necessarily aligned with $\vec{\mathbf{E}}$, therefore the perturbation potential is given by:

$$V = -|\vec{\mathbf{p}}|(E_x \sin\theta\cos\phi + E_y \sin\theta\sin\phi + E_z \cos\theta).$$

1. The unperturbed Hamiltonian $H_0 = \dfrac{\vec{\mathbf{L}}^2}{2I}$ can be solved in angular momentum eigenbasis (or by spherical harmonics). Determine the energy eigenvalues.

2. In order to determine any energy shifts caused by the perturbation V, you have to calculate the matrix elements $< l', m' | V | l, m >$. Show that for any pair of quantum numbers l, m there are only six non-vanishing matrix elements.

3. Calculate the energy shift in first-order perturbation.

4. Calculate the energy shift in second-order perturbation:
$$\Delta_l^{(2)} = \sum_{l' \neq l} \sum_{m'} \frac{|< l', m' | V | l, m >|^2}{E_l - E_{l'}}.$$ Note the special case l=0!

Hint: You may use spherical tensors or spherical harmonics with

$$\sin\theta e^{i\phi}\, Y_l^m = a_{l,m}\, Y_{l+1}^{m+1} - a_{l-1,m-1}\, Y_{l-1}^{m+1}$$
$$\sin\theta e^{-i\phi}\, Y_l^m = -a_{l,-m}\, Y_{l+1}^{m-1} + a_{l-1,m-1}\, Y_{l-1}^{m-1}$$
$$\cos\theta Y_l^m = b_{l,m}\, Y_{l+1}^m + b_{l-1,m}\, Y_{l-1}^m$$

with $$a_{l,m} = \sqrt{\frac{(l+m+1)(l+m+2)}{(2l+1)(2l+3)}}\,;\quad b_{l,m} = \sqrt{\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}}$$