CUA

THE CATHOLIC UNIVERSITY OF AMERICA

Department of Physics
200 Hannan Hall
Washington, D.C. 20064
202-319-5315
Fax 202-319-4448

PhD/MS Comprehensive Examination

FALL 2007

Thursday, October 25, and Friday, October 26, 2007

THE WRITTEN EXAM WILL BE HELD IN ROOM

133 HANNAN HALL

THE EXAM WILL TAKE PLACE:
9:00 AM - 5:00 PM ON THURSDAY October 25, 2007
AND
9:00 AM - 5:00 PM ON FRIDAY, October 26, 2007.

THERE WILL BE A GET TOGTHER ON THE 2ND FLOOR AT 5:00 PM ON FRIDAY.

THE ORAL PORTION OF THE PhD COMPREHENSIVE EXAM WILL BEGIN THE FOLLOWING WEEK.

PLEASE CALL (202) 319-5315) OR STOP BY THE PHYSICS DEPARTMENT OFFICE ON FRIDAY TO INQUIRE ABOUT YOUR SCHEDULED ORAL EXAM.

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MS Comprehensive Examination
Physics Department

Fall 2007

Thursday, October 25 and Friday, October 26, 2007

Room 133 - Hannan Hall

GENERAL INSTRUCTIONS:

This examination is divided into four sections as follows:

Thursday, October 25, 2007

9:00 a.m. - 12:00 Noon	Classical Mechanics - 2 questions
1:00 P.M. - 5:00 P.M.	E & M - 2 questions

Friday, October 26, 2007

9:00 a.m. - 12:00 Noon	Thermodynamics/Stat. Mech. - 2 questions
1:00 P.M. - 5:00 P.M.	Modern Physics/Quantum Mech. - 2 questions

DO EACH PROBLEM IN A SEPARATE BLUE BOOK

PUT YOUR NAME ON EACH BOOK

LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS
FOR EXAMPLE: Mechanics 600-1 or Electromagnetism 500-2

YOU MUST DO TWO (AND ONLY TWO) OF THE QUESTIONS IN EACH AREA.

OF THE TOTAL OF EIGHT QUESTIONS THAT ARE DONE, A MINIMUM OF TWO MUST BE FROM THE 600-LEVEL QUESTIONS.

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Ph.D. Comprehensive Examination

Physics Department

Fall 2007

Thursday, October 25, and Friday, October 26, 2007

Room 133 - Hannan Hall

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9:00 a.m. - 12:00 Noon	Classical Mechanics - 2 questions
1:00 P.M. - 5:00 P.M.	E & M - 3 questions
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9:00 a.m. - 12:00 Noon	Stat Mech. - 2 questions
1:00 P.M. - 5:00 P.M.	Quantum Mech. - 3 questions

DO EACH PROBLEM IN A SEPARATE BLUE BOOK

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LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS
FOR EXAMPLE: Mechanics 600-1

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RULES FOR THE PHYSICS COMPREHENSIVE EXAMINATION

FALL 2007

In order to assure an equitable and fair Comprehensive Examination for all students the following rules will apply to the examination administered by the Department of Physics.

1. The examination will be closed book. Students will not be permitted to bring to the examination room materials such as notes, reference books, calculators, or other aids of any form. The Physics Department will supply copies of the *CRC Mathematical Handbook,* Schaum's *Mathematical Handbook, Table of Functions* by Jahnke and Emde, and the NBS *Handbook of Mathematical Functions*, for the use of those taking the exam should they feel the need for these references during the examination.

   The Physics Department will supply calculators for use during the examination.

   Students who want to use their own mathematical tables and/or handbooks must get permission from the Chairman prior to the beginning of the comprehensive examination. These tables and/or handbooks will remain under the control of the department until after the examination is completed, but will be available to their owners during the exam.

2. If needed, students will be allowed a short break, not to exceed ten minutes, during the examination period. While absent from the examination room students may not consult material or other sources of information concerning any matter pertaining to the examination. Students who leave the examination room will turn in all of their examination materials to the proctor before leaving the room. Examination papers will be returned to the students when they reenter the examination room. Students who fail to return within ten minutes will be considered to have completed that portion of the examination.

   Only one student will be permitted to leave the examination room at a time.

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THE CATHOLIC UNIVERSITY OF AMERICA

Preliminary Examination--Physics Dept.

Friday, October 26, 2007

TIME: 1:00 p.m. - 5:00 p.m.

<u>Room 135 - Hannan Hall</u>

- **YOU <u>MUST</u> DO TWO QUESTIONS IN EACH OF THE AREAS:**

Mechanics

Electricity & Magnetism

Thermodynamics

Modern Physics/Quantum Mechanics

- **DO EACH PROBLEM IN A SEPARATE BLUE BOOK**
- **PUT YOUR NAME ON EACH BLUE BOOK**
- **LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS**
  FOR EXAMPLE: Mechanics #1

M.S. Comprehensive Examination – Fall 2007
Classical Mechanics 500-1

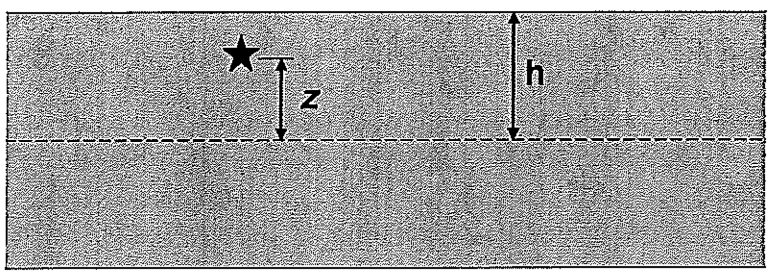
Stars orbit the center of the Galaxy, and to first-order one can describe these orbits as periodic elliptical orbits with foci at the center of mass. However, there is also periodic motion perpendicular to the thin Galactic Plane.

Ignore the orbital motion, and let the shape of the Galaxy be modeled by an infinite thin sheet with half-thickness h and constant mass density ρ_0.

Consider a star located at distance z from the mid-plane of the sheet. If the perpendicular motion of the star is confined to the inside of the Galaxy ($z < h$), show that this motion is periodic, with the angular frequency ω_0 given by

$$\omega_0 = (4\pi G \rho_0)^{1/2}.$$

Hint: In showing this, ignore the orbital motion of the star and consider only the attraction of the mass, and consider the mass distribution in the sheet as analogous to a thin plane of uniform charge density.



M.S. Comprehensive Examination – Fall 2007
Classical Mechanics 500-2

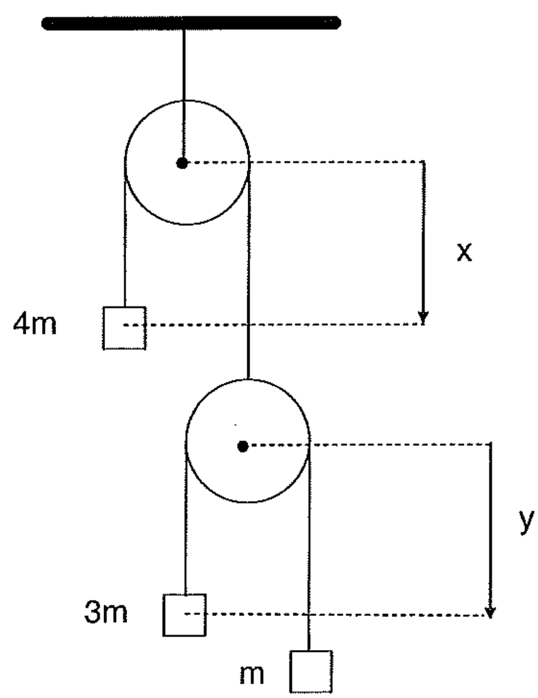
Consider a double Atwood machine constructed as follows: A mass of $4m$ is suspended from a string that passes over a massless, frictionless stationary pulley. The other end of the string supports a similar pulley, over which passes a second string that supports a mass of $3m$ on one end and m on the other. The arrangement is shown in the diagram.

(a) Write out the Lagrangian for this system. Choose as your generalized coordinates those indicated in the diagram, where x is measured from the center of the top pulley to the $4m$ mass and y is measured from the center of the bottom pulley to the $3m$ mass.

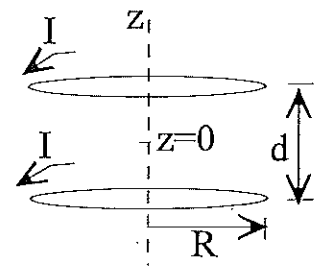
(b) Solve the Lagrange equations to obtain the acceleration of the $4m$ mass.

(c) Now show that the same result for the acceleration of the $4m$ mass is obtained by identifying the relevant forces followed by direct application of Newton's second law. Start by drawing a free body diagram.

(d) Show how your Lagrangian in part (b) is modified if instead the top pulley is a disk of radius R_1 and mass M_1 and the bottom pulley is a disk of radius R_2 and mass M_2.



M.S. Comprehensive Examination – Fall 2007
Electricity and Magnetism 500-1



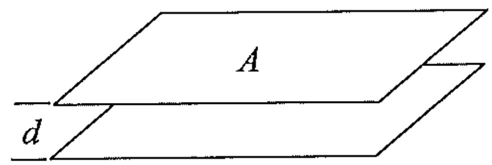
Consider an arrangement of two parallel circular current loops of radius R centered on the same axis and a located a distance d apart. This arrangement, called Helmholtz coils, produces a nearly uniform field in the center.

(a) Find the B field at points along the z axis as a function of z, and show that $\partial B/\partial z$ is zero at the point midway between the coils (z=0).

(b) Determine the separation distance d such that also $\partial^2 B/\partial z^2=0$ at the midpoint, and find the magnetic field at this point.

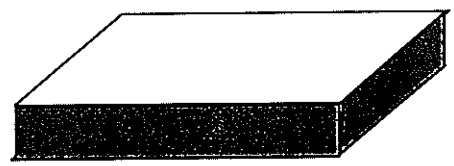
M.S. Comprehensive Examination – Fall 2007
Electricity and Magnetism 500-2

(a) A parallel plate capacitor consists of two plates of area A separated by a narrow gap d. The region between the plates is filled with air (vacuum), and a potential difference V is applied between the plates. Calculate the **capacitance** C_0 of the air-gap capacitor, and the **charge** Q_0 on the positive plate.

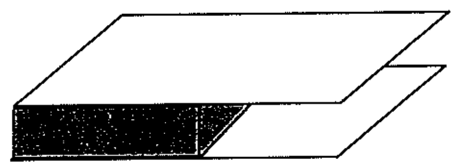


In all the following modifications (b)-(d), the applied potential difference remains V.

(b) The region between the plates is now filled with material of dielectric constant $\kappa > 1$. Calculate the capacitance C and the charge Q (as multiples of C_0 and Q_0 respectively.)



(c) **Half of the area** between the plates is now filled entirely with material of dielectric constant $\kappa > 1$. Calculate the capacitance C and the charge Q (as multiples of C_0 and Q_0 respectively.)



(d) **Half of the gap width** (thickness $d/2$) between the plates is now filled with material of dielectric constant $\kappa > 1$. Calculate the capacitance C and the charge Q (as multiples of C_0 and Q_0 respectively.)

(e) In which of the four cases (a)-(d) is the energy stored in the capacitor the largest?

(f) Discuss the location and magnitude of bound (polarization) charge densities in case (d).

M.S. Comprehensive Examination – Fall 2007
Thermodynamics 500-1

(a) An approximation to the equation of state for a real gas is $(P + a/V^2)(V - b) = NkT$ where a, b, and k are constants. Calculate the work necessary to compress the gas isothermally from V_1 to $V_2 < V_1$.

(b) Measurements show that nitrogen gas obeys the van der Waals equation of state with constant $b = 3.94 \times 10^{-5}\ m^3/mol$. What is the size of the nitrogen molecule?

M.S. Comprehensive Examination – Fall 2007
Thermodynamics 500-2

(a) Prove the Maxwell relations

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V \quad \text{and} \quad \left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p$$

(b) Use these to show that $\left(\frac{\partial C_V}{\partial V}\right)_T = T\left(\frac{\partial^2 p}{\partial T^2}\right)_V$ and $\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V - p$

(c) Prove that the Joule-Thompson coefficient μ is given by

$$\mu = \left(\frac{\partial p}{\partial T}\right)_H = \frac{V}{C_p}(T\alpha - 1)$$

where α is the volume thermal expansion coefficient. Show that μ is zero for an ideal gas.

Now consider a gas that obeys the van der Waals equation of state, $p = \frac{RT}{v-b} - \frac{a}{v^2}$, where v is the molar volume.

(d) Show that C_V is independent of V for this gas.

(e) Find $\left(\frac{\partial U}{\partial V}\right)_T$ for this gas. Explain how this result implies that the parameter a is a measure of the inter-particle interactions.

M.S. Comprehensive Examination – Fall 2007
Quantum Mechanics/Modern Physics 500-1

Consider a particile with total energy E that is moving in the $+x$ direction, and is subject to a step-function potential $V(x)$ as shown below:

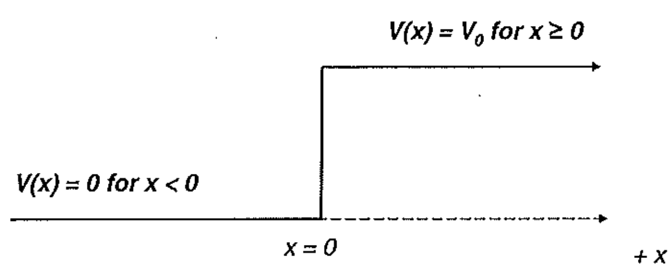
$$V(x) = 0 \qquad x < 0$$
$$V(x) = V_0 \qquad x \geq 0 \quad \text{(with } V_0 > 0.)$$

Consider the case where $E > V_0$.

(a) Write the general form of the wave functions in the two regions.

(b) Derive the transmission and reflection coefficients for the barrier, assuming the particle is incident from $x < 0$ and moving in the $+x$ direction.

(c) How do the resulting predictions for transmission and reflection at the barrier differ from those of classical mechanics?



M.S. Comprehensive Examination – Fall 2007
Quantum Mechanics/Modern Physics 500-2

An electron is in the spin state $\frac{1}{3}|+\rangle + \frac{2\sqrt{2}}{3}|-\rangle$, where the symbols $|+\rangle$ and $|-\rangle$ represent the states with $m_s = +½$ and $-½$ respectively.

(a) If you measure S_z, the z-component of the spin angular momentum, what values can you obtain, and what is the probability of obtaining each one?

(b) Find $\langle S_z \rangle$, the expectation value of S_z.

(c) Find the rms (root-mean-square) uncertainty in S_z.

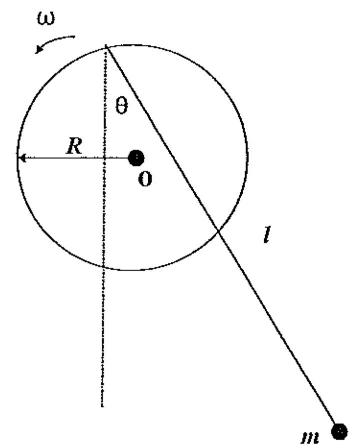
The states of the electron in the hydrogen atom can be labeled $|n, l, m_l, \pm\rangle$, where n is the principal quantum number, l and m_l specify the orbital angular momentum, and ± labels the spin projection as above. At one instant, the electron in a hydrogen atom is in the state

$$|\psi\rangle = \frac{1}{\sqrt{5}}|2,1,-1,+\rangle + \frac{2}{\sqrt{5}}|1,0,0,-\rangle .$$

d) Find the expectation values of the following operators for this state:

H (in units of the ground-state energy $E_1 = -13.6$ eV)
L^2
L_z
S^2
S_z

Ph.D. Comprehensive Examination – Fall 2007
Classical Mechanics 600-1



Consider a simple pendulum with a mass m at the end of a massless rod of length l. The rod is attached to a pivot point on the rim of a rotating wheel with radius **R**, which rotates at a constant rotational velocity ω about the horizontal axis **O**. The rod pivots freely in the vertical plane, making an instantaneous angle θ with the vertical. Gravity is exerting a force in the downward direction (corresponding to $\theta = 0$). At time $t = 0$, the pivot point is at the top of the wheel.

(a) Derive in Cartesian coordinates the expressions for velocity and acceleration of the mass m.

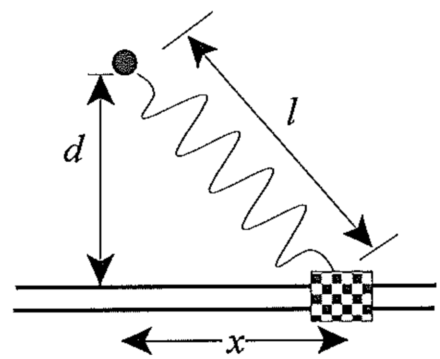
(b) Write the resulting Lagrangian for this system in terms of the generalized coordinate θ.

(c) What is the equation of motion for $\ddot{\theta}$?

(d) Suppose that instead of a solid rod, the mass **m** is attached to an end of a spring with a spring constant **k**. The equilibrium length of the spring is l_0.
Write the Lagrangian for this system. (You don't have to solve it!)

Ph.D. Comprehensive Examination – Fall 2007
Classical Mechanics 600-2

A cart of mass m slides on a straight horizontal rail. The cart is tied by a spring to a pole at distance d from the rail. The spring has an unstretched length l_0 and spring constant k. Assume that the spring is massless and that the cart can be approximated by a mass point moving without friction on the rail.



(a) Find the potential energy as function of the cart's position x on the rail. ($x = 0$ when the cart is directly below the fixed end of the spring.)

(b) Find the equilibrium position(s) of the cart for the two cases $d \geq l_0$ and $d < l_0$.

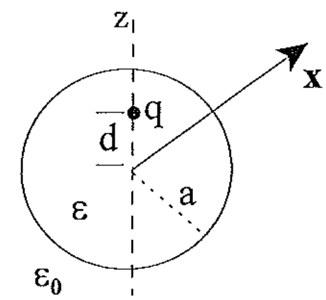
(c) For each case, expand the potential around the equilibrium position(s).

(d) For each case, find the approximate frequency of oscillations of the cart.

(e) Consider the special cases that $l_0 = d$ and $l_0 = 0$ and show that for one extreme the motion is purely harmonic and that for the other extreme the quadratic term in the potential vanishes.

Ph.D. Comprehensive Examination – Fall 2007
Electricity and Magnetism 600-1

A point charge q is embedded inside a solid dielectric sphere of radius a and electric permittivity ε, at a distance d from the center of the sphere. The sphere is positioned in free space (permittivity ε_0).



(a) The potential at any point can be expressed as

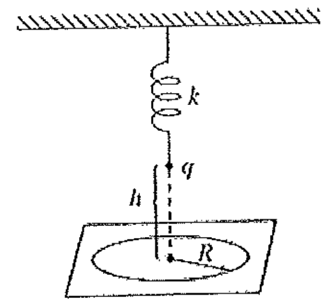
$$\Phi(\mathbf{x})=\frac{1}{4\pi\varepsilon}\frac{q}{|\mathbf{x}-\mathbf{d}|}+\Phi_1(\mathbf{x})$$

where $\Phi_1(\mathbf{x})$ satisfies Laplace's equation. Determine the potential inside and outside the sphere by expanding in terms of Legendre polynomials.

(b) If q is at the center of the sphere (d=0), determine **E**, **D** and **P** inside the sphere. What is the polarization charge density ρ_p and the surface polarization charge density σ_p?

Ph.D. Comprehensive Examination – Fall 2007
Electricity and Magnetism 600-2

A particle of mass m and charge q is attached to a spring with force constant k, hanging from the ceiling as shown in the figure. Its equilibrium position is a distance h above the floor. It is pulled down a distance d below equilibrium and released, at time $t=0$. The Poynting vector time-averaged over a complete cycle is given as:



$$< S >= \frac{\mu_0}{c} \frac{p_0^2 \omega^4}{32\pi^2} \left(\frac{\sin^2 \theta}{r^2} \right) \hat{r}$$

(a) Under the usual assumptions ($d << \lambda << h$), calculate the intensity of the radiation hitting the floor, as a function of the distance R from the point directly below q. The intensity here is the average power per unit area of floor.

(b) At what R is the radiation most intense? Neglect the radiative damping of the oscillator.

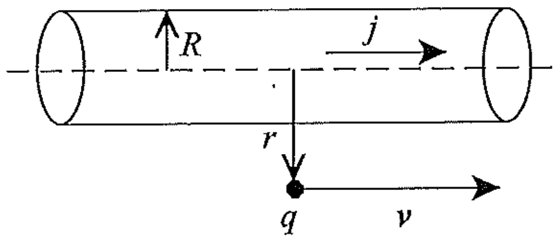
(c) As a check on your formula, assume the floor is of infinite extent, and calculate the average energy per unit time striking the entire floor. Is it what you would expect?

(d) Because it is losing energy in the form of radiation, the amplitude of the oscillation will gradually decrease. After at what time t has the amplitude been reduced to d/e? Assume the fraction of the total energy lost in one cycle is very small.

(e) Roughly sketch the intensity profile of the oscillating dipole.

Ph.D. Comprehensive Examination – Fall 2007
Electricity and Magnetism 600-3

An infinitely long cylindrical wire of radius R carries a constant current density $\vec{j} = j\hat{z}$ in the $+z$ direction. The charge is carried by equal densities of positive and negative charges in the $\pm z$



directions, so the net charge density ρ is zero. A positive point charge q is moves with velocity v parallel to the current, at a distance $r > R$ from the axis of the wire.

(a) Find the *magnetic* force F_m experienced by the moving charge in the laboratory frame.

(b) Consider the current density and charge density as the components of a 4-vector current density $j_\alpha = (c\rho, \vec{J}) = (c\rho, j_x, j_y, j_z)$ with $\alpha = 0,1,2,3$. In the laboratory system, $j_\alpha = (0; 0, 0, j)$.
Perform a Lorentz transformation to find the current density 4-vector j'_α in the rest frame of the point charge q.

(c) Show that in the rest frame of the charge, the stationary charge experiences an *electric* field and consequently an electric force F'_e. In the limit $v \ll c$, find the magnitude and direction of this force, and show that it agrees with the magnetic force calculated in (a). (The restriction $v \ll c$ is not necessary, but when v is large one must consider time dilation in the definition of the force in order to make the results in the two frames agree exactly. You may ignore this effect.)

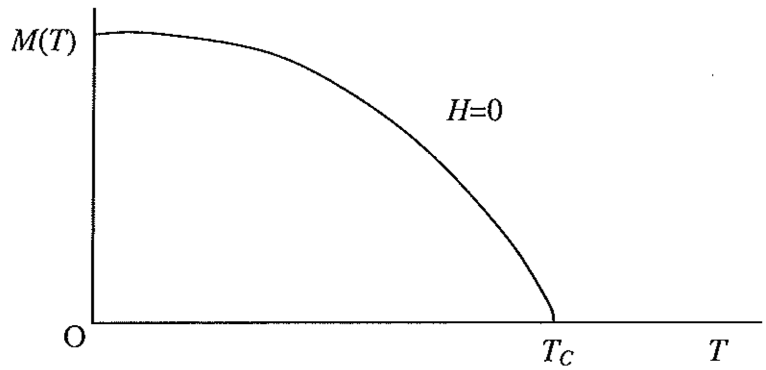
(d) Write the equation of continuity (charge conservation), and show that it can be written in the covariant form $\partial_\alpha J^\alpha = 0$.

(e) The vector and scalar potentials $\vec{A}$ and Φ can be combined into a 4-vector potential $A^\alpha = (\Phi, \vec{A}) = (\Phi, A_x, A_y, A_z)$ in Gaussian units (or $A^\alpha = (\Phi/c, \vec{A})$ in SI units.)
Using the relationship between the potentials and the electric and magnetic fields, show that all the components of $\vec{E}$ and $\vec{B}$ can written (up to a factor of c, depending on the system of units) in the form of elements of a contravariant, second-rank, antisymmetric field-strength tensor

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \quad \text{(where } \partial^0 = \partial_0 \text{ but } \partial^1 = -\partial_1, \ldots).$$

Write out this tensor explicitly in terms of the components of $\vec{E}$ and $\vec{B}$.

Ph.D. Comprehensive Examination – Fall 2007
Statistical Mechanics 600-1



For a ferromagnetic material in the absence of an applied field (*H*=0), the spontaneous magnetization is a maximum at *T*=0, decreases to zero at the critical temperature $T = T_C$, and is zero for all $T > T_C$. For temperatures just below T_C the magnetic susceptibility and the temperature coefficient of *M* might be modeled by the expressions

$$X_T = \left(\frac{\partial M}{\partial H}\right)_T = \frac{a}{(1 - T/T_C)} + 3bH^2$$

$$\left(\frac{\partial M}{\partial T}\right)_H = \frac{1}{T_C}\frac{f(H)}{\left(1-\frac{T}{T_C}\right)^2} - \frac{M_0}{2T_C}\frac{1}{\left(1-\frac{T}{T_C}\right)^{1/2}}$$

where M_0, T_C, *a* and *b* are constants and *f*(*H*) is a function of *H* alone with the property that $f(H=0) = 0$.

(a) Find *f*(*H*) by using the fact that *M* is a state function.

(b) Find *M*(*H*,*T*).

Ph.D. Comprehensive Examination – Fall 2007
Statistical Mechanics 600-2

(a) Consider an ideal gas of N electrons in a volume V. Calculate the Fermi energy ε_F in terms of N and V.

Consider now a simple model of a metal that consists of a crystal lattice made up of N particles in a volume V and an ideal gas of N electrons confined to the same volume V. In this model, all interactions between the electrons and between the electrons and the lattice are neglected. The specific heat C_V of the metal may be written as the sum of the lattice and electron gas contributions: $C_V = C_V^L + C_V^e$.

(b) The following argument leads to a result that is inconsistent with experimental observations:
"*By equipartition of energy, at room temperature* $C_V^L = 3Nk$ *and* $C_V^e = 3Nk/2$*, therefore the metal has a specific heat that is ~50% larger than that of an insulator (the crystal lattice with no free electron gas)*".

Explain why this argument is incorrect.

(c) Make a rough estimate of the width of the Fermi function at a temperature T, where T is of the order of room temperature. Use this result to make a rough, but much improved, estimate of C_V^e. Explain how your result resolves the argument posed in part (b).

We now wish to make a more quantitative calculation of C_V^e for low temperatures ($T \ll T_F$, where T_F is the Fermi temperature), good to second order in T. (See hint below.)

(d) Calculate the chemical potential $\mu(T, V, N)$ of the free-electron gas to order T^2.

(e) Find the mean energy per particle $\langle \varepsilon \rangle$ of the free-electron gas to order T^2. Eliminate μ from your result using the result from part (d). Show that you recover the familiar result $\langle \varepsilon \rangle = \frac{3}{5}\varepsilon_F$ at $T = 0$.

(f) Find C_V^e in this same approximation. Discuss the relative contributions of C_V^L and C_V^e to C_V as $T \to 0$.

<u>**Hint:**</u> The following result may be useful for parts (d) and (e):

$$\langle \varepsilon^j \rangle = \frac{3}{2\varepsilon_F^{3/2}} \int_0^\infty \frac{\varepsilon^{j+1/2} d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1} \approx \frac{3}{2j+3}\frac{\mu^{j+3/2}}{\varepsilon_F^{3/2}} + \left(j+\frac{1}{2}\right)\frac{\pi^2}{4}\frac{\mu^{j-1/2}(kT)^2}{\varepsilon_F^{3/2}}$$

Ph.D. Comprehensive Examination – Fall 2007
Quantum Mechanics 600-1

Two ions of equal mass m and electric charges q_1 and q_2 interact through harmonic forces described by the potential

$$V(\vec{r}_1,\vec{r}_2) = \frac{m\omega^2}{2}(\vec{r}_1 - \vec{r}_2)^2,$$

which approximates the total interaction between the ions near equilibrium.
The system is subject to a uniform electric field $\mathscr{E}$.

(a) Set up the Hamiltonian of the system in its center-of-mass (c.m.) frame using the c.m. operators $\vec{R}$ and $\vec{P}$ and the relative operators $\vec{r} = \vec{r}_1 - \vec{r}_2$ and $\vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)$.

(b) Find the energy eigenvalues for the relative part of the system. These energy eigenvalues have an electric-field dependent part, which can be written as $E_{el} = -\frac{\alpha}{2}\mathscr{E}^2$. What is the coefficient α, i.e. the electric polarizability of the system?

(c) Find the Heisenberg equations of motion for the c.m. operators $\vec{R}$ and $\vec{P}$ and the relative operators $\vec{r}$ and $\vec{p}$.

(d) The electric-field dependent terms in the Hamiltonian can be written as $H' = \vec{D}\cdot\vec{\mathscr{E}}$. Find the electric dipole moment $\vec{D}=\vec{D}_{cm}(\vec{R})+\vec{d}_{rel}(\vec{r})$ and determine the time-dependent expectation values of the c.m. part and the relative part of the dipole moment and relate them to α.

Ph.D. Comprehensive Examination – Fall 2007
Quantum Mechanics 600-2

A quantum mechanical system is described by the Hamiltonian $H = H_0 + H'$, where $H' = i\lambda[A, H_0]$ is a perturbation on the unperturbed Hamiltonian H_0. A is a Hermitian operator and λ is a real number.
Let B be a second Hermitian operator and let $C = i[B, A]$.

(a) Use first order perturbation theory to show that, to first order in λ, $\langle B\rangle \approx \langle B\rangle_0 + \lambda\langle C\rangle_0$, where $\langle B\rangle$ is the expectation value of B in the perturbed ground state and the expectation values of the operators A, B, and C in the unperturbed and non-degenerate ground states are $\langle A\rangle_0$, $\langle B\rangle_0$, and $\langle C\rangle_0$, respectively.

(b) Apply this result to the case where

$$H_0 = \sum_{i=1}^{3}\left(\frac{p_i^2}{2m} + \frac{1}{2}m\omega^2 x_i^2\right) \text{and } H' = \lambda x_3$$

and compute the ground state expectation values $\langle x_1\rangle$, $\langle x_2\rangle$, and $\langle x_3\rangle$ to first order in λ. Start by identifying the operator A.

(c) Find the exact ground state wave function for the Hamiltonian $H = H_0 + H'$ in part (b) and from it find the exact values for $\langle x_1\rangle$, $\langle x_2\rangle$, and $\langle x_3\rangle$. Compare your results to those obtained in part (b).

Hints:

$$[x^n, p] = i\hbar n x^{n-1}$$

The ground state wavefunction for the harmonic oscillator is

$$\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$

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The components of the total angular momentum operator $\vec{J}$ satisfy the commutation relations $[J_x, J_y] = i\hbar J_z$, etc. The ladder operators J_+ and J_- are defined by $J_\pm = J_x \pm J_y$. With this definition, it can be shown that $J_\pm |j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm1)}\,|j,m\pm1\rangle$. Similar ladder operators can be defined for orbital angular momentum $\vec{L}$ and spin angular momentum $\vec{S}$.

(a) Find the products J_+J_- and J_-J_+, and show that

$$J^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_z^2 .$$

Consider an electron (spin $s = ½$) in a p-state of orbital angular momentum ($l = 1$). The total angular momentum of the electron is $\vec{J} = \vec{L} + \vec{S}$.

(b) List all possible states of this system in the notation $|m_l, m_s\rangle$ (where m_l and m_s are the quantum numbers specifying the eigenstates of L_z and S_z; the quantum numbers l and s are omitted, as they are the same for all states.) Write the value of $m = m_l + m_s$ for each state.

(c) Explicitly evaluate the quantities $S_+|m_l, m_s\rangle$, $S_-|m_l, m_s\rangle$, $L_+|m_l, m_s\rangle$, and $L_-|m_l, m_s\rangle$ for each of the states in (b).

(d) Writing the J^2 operator in terms of L_+, L_-, S_+, S_- and J_z, evaluate $J^2|m_l, m_s\rangle$ explicitly for the following states listed in (b):
1) the state(s) with $m = +3/2$.
2) the state(s) with $m = +1/2$.
Which of these states (if any) are eigenfunctions of J^2?

(e) Consider the subspace of states in (b) for which $m = m_l + m_s = +½$. Show that the J^2 operator can be represented by a matrix operating on these states. Write the matrix and find its eigenvalues.

(f) Check that your result in (e) is consistent with the rules for angular momentum addition.