CUA

THE CATHOLIC UNIVERSITY OF AMERICA

Department of Physics
200 Hannon Hall
Washington, D.C. 20064
202-319-5315
Fax 202-319-4448

RULES FOR THE PHYSICS COMPREHENSIVE EXAMINATION

Fall 2004

In order to assure an equitable and fair Comprehensive Examination for all students the following rules will apply to the examination administered by the Department of Physics.

1. The examination will be closed book. Students will not be permitted to bring to the examination room materials such as notes, reference books, calculators, or other aids of any form. The Physics Department will supply copies of the *CRC Mathematical Handbook,* Schaum's *Mathematical Handbook, Table of Functions* by Jahnke and Emde, and the NBS *Handbook of Mathematical Functions*, for the use of those taking the exam should they feel the need for these references during the examination.

   Students who want to use their own mathematical tables and/or handbooks must get permission from the Chairman prior to the beginning of the comprehensive examination. These tables and/or handbooks will remain under the control of the department until after the examination is completed, but will be available to their owners during the exam.

2. If needed, students will be allowed a short break, not to exceed ten minutes, during the examination period. While absent from the examination room students may not consult material or other sources of information concerning any matter pertaining to the examination. Students who leave the examination room will turn in all of their examination materials to the proctor before leaving the room. Examination papers will be returned to the students when they reenter the examination room. Students who fail to return within ten minutes will be considered to have completed that portion of the examination.

   <u>Only one student will be permitted to leave the examination room at a time.</u>

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Preliminary Examination--Physics Dept.

Fall 2004

Thursday, October 28, and Friday, October 29, 2004

Room 135 - Hannan Hall

- **YOU MUST DO TWO QUESTIONS IN EACH OF THE AREAS:**

Thursday, October 28, 2004

Mechanics

Electricity & Magnetism

Friday, October 29, 2004

Thermodynamics

Modern Physics/Quantum Mechanics

- **DO EACH PROBLEM IN A SEPARATE BLUE BOOK**

- **PUT YOUR NAME ON EACH BLUE BOOK**

- **LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS**
  FOR EXAMPLE: Mechanics #1

CUA

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Department of Physics
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Washington, D.C. 20064
202-319-5315
Fax 202-319-4448

MS Comprehensive Examination
Physics Department

Fall 2004

Thursday, October 28 and Friday, October 29, 2004

Room 135 - Hannan Hall

GENERAL INSTRUCTIONS:

This examination is divided into four sections as follows:

Thursday, October 28, 2004

9:00 a.m. - 12:00 Noon	Classical Mechanics - 2 questions
1:00 P.M. - 5:00 P.M.	E & M - 2 questions

Friday, October 29, 2004

9:00 a.m. - 12:00 Noon	Thermodynamics/Stat. Mech. - 2 questions
1:00 P.M. - 5:00 P.M.	Modern Physics/Quantum Mech. - 2 questions

DO EACH PROBLEM IN A SEPARATE BLUE BOOK

PUT YOUR NAME ON EACH BOOK

LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS
FOR EXAMPLE: Mechanics 600-1 or Electromagnetism 500-2

YOU MUST DO TWO (AND ONLY TWO) OF THE QUESTIONS IN EACH AREA.

OF THE TOTAL OF EIGHT QUESTIONS THAT ARE DONE, A MINIMUM OF TWO MUST BE FROM THE 600-LEVEL QUESTIONS.

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Department of Physics
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Washington, D.C. 20064
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Fax 202-319-4448

Ph.D. Comprehensive Examination

Physics Department

Fall 2004

Thursday, October 28, and Friday, October 29, 2004

Room 135 - Hannan Hall

GENERAL INSTRUCTIONS:

This examination is divided into four sections as follows:

<u>Thursday, October 28, 2004</u>

9:00 a.m. - 12:00 Noon	Classical Mechanics - 2 questions
1:00 P.M. - 5:00 P.M.	E & M - 3 questions

<u>Friday, October 29, 2004</u>

9:00 a.m. - 12:00 Noon	Stat Mech. - 2 questions
1:00 P.M. - 5:00 P.M.	Quantum Mech. - 3 questions

<u>DO EACH PROBLEM IN A SEPARATE BLUE BOOK</u>

PUT YOUR NAME ON EACH BOOK

LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS
FOR EXAMPLE: Mechanics 600-1

A spherical asteroid of uniform density ρ and radius R spins about its central axis at an initial speed of one rotation per minute. Its center of mass is at rest in interstellar space.

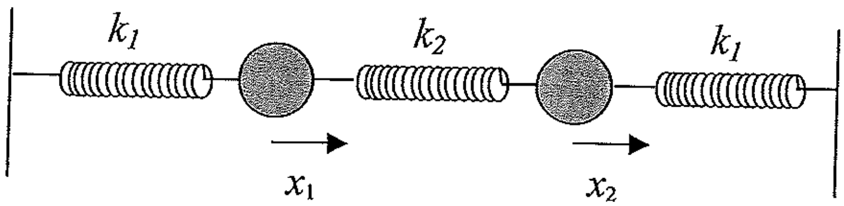
The asteroid gradually acquires meteoric material of the same density, until a few billion years later, its radius has doubled.

(a) If, on average, this material has arrived radially, what is the asteroid's final rate of rotation?

Hint: Consider what is conserved in fully inelastic collisions. The moment of inertia for a solid sphere is $I = 2MR^2/5$).

(b) If the arriving material had negligible velocity before colliding with the asteroid, how much kinetic energy was dissipated by inelastic collisions during the process?

Two equal masses m are connected by three springs as indicated, with a fixed wall on each side.



The springs have spring constants k_1 and k_2 as indicated in the diagram. Assume that the masses move in the horizontal (x) direction only, with displacements x_1 and x_2.

(a) Write equations of motion for each mass, in the form $\frac{d^2 x_i}{dt^2} = \ldots.$

(b) Show that the equations of motion are separated by defining two other variables,
$u_1 = x_1 + x_2$ and $u_2 = x_1 - x_2$.
Find the equations of motion for u_1 and u_2.

(c) Solve these equations of motion for u_1 and u_2. Identify the frequencies of oscillation associated with each of these.

(d) At time $t = 0$, mass 1 is held fixed but mass 2 is displaced such that:

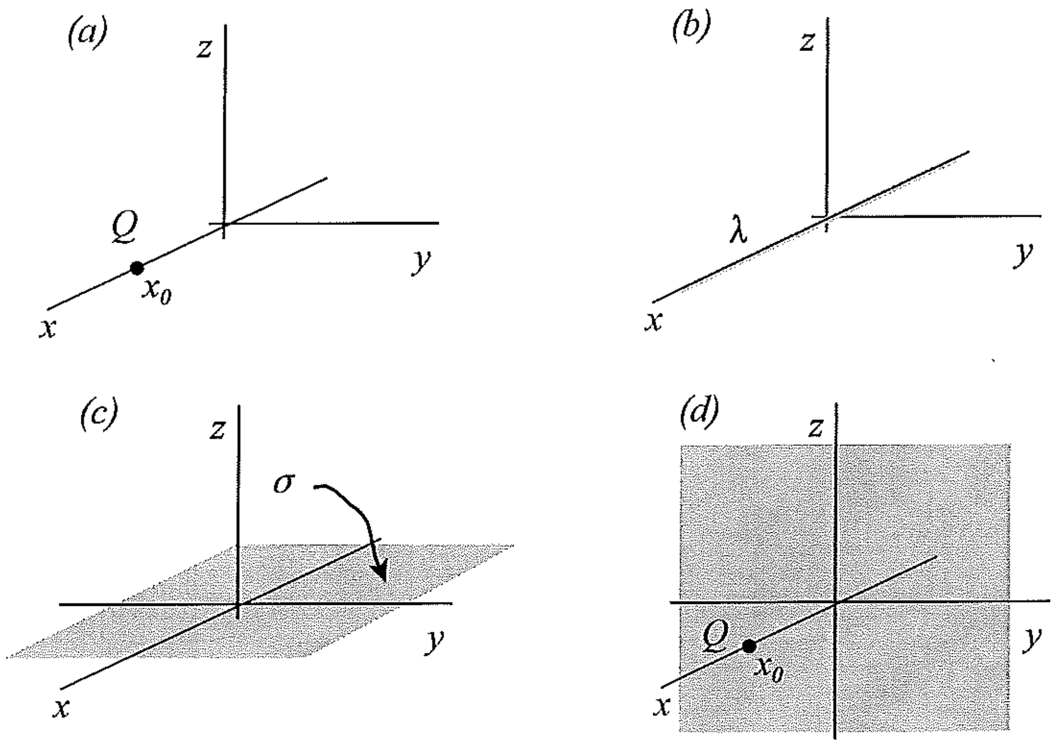
$$x_1 = 0, \quad \dot{x}_1 = 0, \quad x_2 = 1, \quad \dot{x}_2 = 0.$$

Construct solutions for $x_1(t)$ and $x_2(t)$.

MS Comprehensive/Preliminary Examination
Fall 2004
Electricity and Magnetism: EM 500-1

Calculate both the electric potential and the electric field for the following charge distributions:

(a) a point charge Q at a position x_0 on the x-axis as shown.
(b) a uniform line charge, λ = charge per unit length, lying along the x-axis from $-\infty$ to ∞.
(c) a uniform surface charge, σ = charge per unit area, distributed on the x-y plane.
(d) a point charge Q on the x-axis at x_0 and the y-z plane is a conductor.



Consider a line of charge that lies along the z-axis from $-a$ to $+a$. The linear charge density (charge per unit length) is given by

$$\lambda = \begin{cases} -q/a & (-a \le z \le 0) \\ +2q/a & (\;0 \le z \le a) \end{cases}.$$

(a) Determine the monopole moment of the charge distribution and its contribution to the potential at an arbitrary point $\vec{r}$, where $r = |\vec{r}| >> a$.

(b) Determine the dipole moment of the distribution and its contribution to the potential at an arbitrary point $\vec{r}$, where $r = |\vec{r}| >> a$.

(c) If a charge q' is brought from infinity to a point $x=0, y=0, z=50a$, how much work is done?

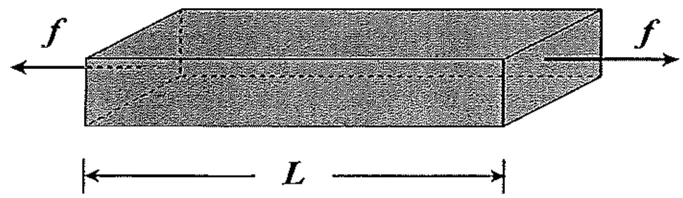
(d) If a charge q' is brought from infinity to a point $x=50a, y=0, z=0$, how much work is done?

(e) Suppose a uniform electric field parallel to the z-axis, $\vec{E} = \hat{z}\,E_0$, is imposed on the distribution. Calculate the interaction energy of this field with the charge distribution. Calculate the force that it exerts on the distribution. Calculate the torque that it exerts on the distribution.

A building at a temperature T is heated by means of a heat pump, which uses a river at T_0 as a source of heat. The heat pump, which may be assumed to have an ideal performance, consumes energy at a constant rate P, and the building loses heat to its surroundings at a rate $\alpha(T - T_0)$, where α is a constant. Show that the equilibrium temperature of the building, T_e, is given by

$$T_e = T_0 + \frac{P}{2\alpha}\left[1 + \left(1 + \frac{4\alpha T_0}{P}\right)^{1/2}\right].$$

A piece of rubber of length L and volume V is subjected to work done by hydrostatic pressure p and by a tensional force f.



(a) Construct the expression for dU, where U is the internal energy.
(b) Generate the thermodynamic potentials which have as proper variables (S, V, f), (S, p, f) and (T, p, f), where S is entropy and T is temperature.
(c) Derive the Maxwell relations

$$\left(\frac{\partial T}{\partial V}\right)_{L,S} = -\left(\frac{\partial p}{\partial S}\right)_{V,L},$$

$$\left(\frac{\partial S}{\partial L}\right)_{T,p} = -\left(\frac{\partial f}{\partial T}\right)_{p,L},$$

and

$$\left(\frac{\partial S}{\partial f}\right)_{p,L} = +\left(\frac{\partial L}{\partial T}\right)_{S,p}.$$

Consider a particle of mass m confined by a one-dimensional potential well of the form

$$V(x) = V_0(x) + \upsilon(x).$$

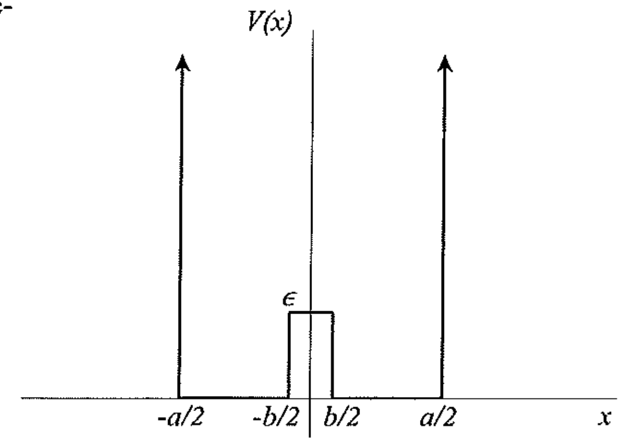
Here

$$V_0(x) = \begin{cases} \infty & (|x| > a/2) \\ 0 & (|x| \le a/2) \end{cases}$$

and

$$\upsilon(x) = \begin{cases} 0 & (|x| > b/2) \\ \epsilon & (|x| \le b/2) \end{cases}$$

with $b < a$.



(a) Initially ignore $\upsilon(x)$ and treat the problem as if $V = V_0$. Calculate the eigenfunctions ψ_n and the associated energy eigenvalues E_n.

(b) Treating the term $\upsilon(x)$ in the potential as small, use first-order time-independent perturbation theory to obtain the corrections to the energy eigenvalues.

(c) Examine these corrections in the limit $b \ll a$. Expand the corrections in powers of (b/a) and keep only the first non-vanishing terms.

(d) Give a physical interpretation of the different behavior for odd and even values of n.

The Bohr model of a hydrogen atom envisions an electron (mass = m_e, charge = - e) moving in a circular orbit around a nucleus of charge q. The interaction between them is the Coulomb force

$$F = -\frac{1}{4\pi\epsilon_0}\frac{qe}{r^2}.$$

For hydrogen the nucleus is a proton (mass = m_p = 1836 m_e, charge = + e). The allowed orbits are determined by the quantum condition on the angular momentum

$$L = n\hbar \qquad (n = 1, 2, 3 \ldots)$$

(a) Derive expressions for the radii r_n and energies E_n of the allowed orbits.

Consider the following modifications to this model and determine quantitatively the effect on r_n and E_n in each case.

(b) Muonium: the electron in the hydrogen atom is replaced by a muon – same charge but the mass is $m_\mu = 207\ m_e$.

(c) Positronium: the proton in the hydrogen atom is replaced by a positron (mass = m_e and charge = + e).

(d) Singly-ionized helium: the proton in the hydrogen atom is replaced by a helium nucleus composed of two protons and two neutrons (you may take the nuclear mass = 4 m_p for the purposes of this problem).

Consider a bob of mass m attached to a massless rod of length a, which is pivoted from the ceiling such that the bob can swing in any direction.

(a) Determine the Lagrangian for this problem and find the constants of the motion.

(b) Solve the problem for the case where the motion is confined to a vertical plane.

(c) Returning to the general case, use separation of variables to obtain a function $t(\theta)$. Solve the elliptic integral for small θ (by substituting α for θ^2) and find $\theta^2(t)$ and $\varphi(t)$.

(d) Find the limiting angles θ_{min} and θ_{max} between which the bob precesses.

Consider a charged particle moving in a uniform magnetic field $\vec{B} = B\hat{z}$.

Show that the magnetic moment $\vec{\mu} = \dfrac{q\vec{L}}{2m}$ is an adiabatic invariant by carrying out the following steps:

(a) Obtain the Hamiltonian and Hamilton's equations of motion for the particle (using cylindrical coordinates: r, ϕ, z).

(b) Consider the case where the particle is circling around the lines of force, i.e. the steady motion solution, and find the frequency of this motion.

(c) Argue that the action integral $J_\phi = \oint p_\phi \, d\phi$ is an adiabatic invariant, and relate it to $\vec{\mu}$.

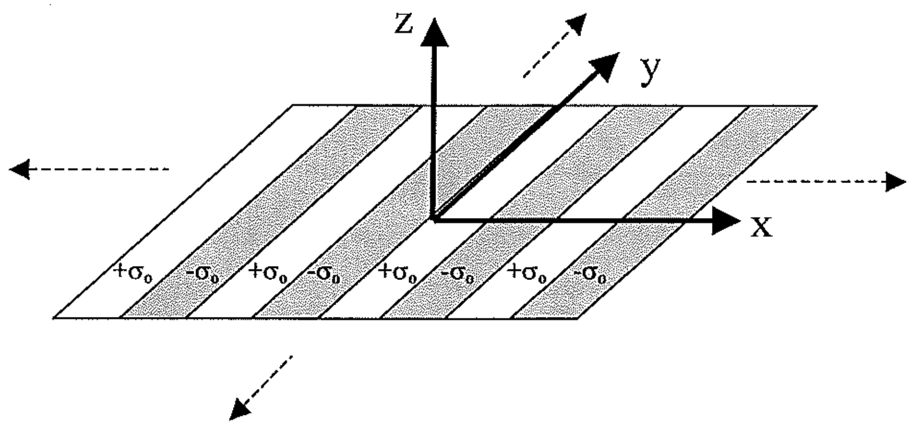
Hints:

(i) Use the vector potential $\vec{A} = -\frac{1}{2}\vec{r} \times \vec{B}$ for the uniform magnetic field $\vec{B}$.

(ii) You may begin with the Lagrangian $L = \frac{1}{2} m v^2 + q\vec{A} \cdot \vec{v}$ and derive the Hamiltonian, or if you know its form, you may write the Hamiltonian directly.

A very thin insulating sheet covers the entire *x-y* plane. The sheet is divided in strips of width L and infinite length parallel to the *y*-axis.

The strip extending from $x = 0$ to $x = L$ carries surface charge density $+\sigma_o$. The neighboring strip, extending from $x = L$ to $x = 2L$, holds an equal and opposite charge density $-\sigma_o$. This pattern repeats with strips carrying $\pm\sigma_o$ alternating along the plane in both directions to infinity, as indicated in the diagram.



Find the electrostatic potential everywhere in space, both above and below the *x-y* plane, by carrying out the following steps:

(a) Write down Laplace's equation, $\nabla^2\Phi = 0$ in appropriate coordinates and perform a separation of variables. [Hint: the electrostatic potential is a function of only two spatial variables.]

(b) List all the boundary conditions that govern the potential at $z = 0$, $z >> 0$, and $z << 0$.

(c) Write out appropriate expansions for the electrostatic potential both above and below the *x-y* plane. [Hint: take into account the symmetry requirements $\Phi(x,z) = \Phi(x,-z)$ and $\Phi(x+2L,z) = \Phi(x,z)$.]

(d) Verify that the terms in your expansion satisfy Laplace's equation.

(e) Use the boundary conditions and symmetry requirements to solve for the electrostatic potential both above and below the *x-y* plane. Be sure to solve explicitly for the unknown coefficients in your expansions.

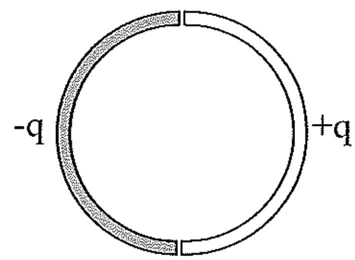
An isolated, uniformly magnetized sphere of radius a has constant magnetization $\vec{M} = M_0\,\hat{z}$ inside the sphere.

(a) State clearly the mathematical boundary conditions on the $\vec{B}$ and $\vec{H}$ fields on the surface of the uniformly magnetized sphere.

(b) Determine the $\vec{B}$ and $\vec{H}$ fields inside the sphere (region 1: $r < a$) and outside the sphere (region 2: $r > a$).

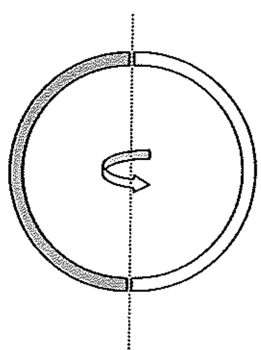
(c) Now consider another case: a sphere of magnetic permeability μ is placed in a uniform magnetic field $\vec{B} = \vec{B}_0$ oriented along the z-axis. Show that in this case the magnetization of the sphere is again constant and find its magnitude.

A thin **circular ring** of radius R is composed of two semi-circles, each made of an insulating material. The two halves hold total charges +/- q respectively, distributed uniformly on each side.



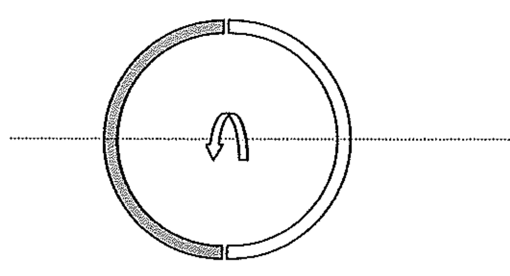
(a) What is the dipole moment of the ring?

(b) The ring spins around a central axis parallel to the plane of the ring, as shown, with angular velocity ω. How much power is radiated away?



(c) If the ring starts with initial angular velocity ω_o, how will its angular velocity drop with time? (The moment of inertia for a ring spinning on such an axis is $\frac{1}{2} M R^2$.)

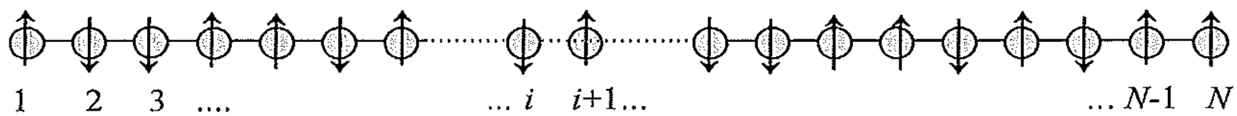
(d) If the ring rotates around the axis shown in the figure below, would the power radiated be the same as in part (b)? Please explain your answer in words only; no calculations are required.



(e) For comparison, consider two charged particles, positive and negative, connected by an insulating linear spring. The system is set oscillating and radiates away energy. Will the frequency of oscillation change over time? Will the amplitude of oscillation change over time? Explain your answer in words; no calculations are required.

PhD/MS Comprehensive Examination
Fall 2004
Statistical Mechanics: SM600-1

Consider a system of N particles placed along a one dimensional lattice, each particle being separated from its neighbor by a distance a. The particles can exist in one of two states: spin up (↑) or spin down (↓). One of the possible configurations of the lattice is shown below. Assume that $N \gg 1$.



By virtue of their spins each of the particles possesses a magnetic moment μ that is aligned with the direction of the spin.

Part 1

Suppose that the spins are non-interacting so that their directions are independent of one another (that is the direction of a given spin–up or down–has no influence on the orientation of any other spin).

(a) Write an expression for $P(n)$, the probability that exactly n of the spins are in the "up" state, if p is the probability that any given spin is in the "up" state. Note that p is not necessarily = ½.

(b) What is n_{max}, the most probable value of n? What is $<n>$, the average value of n?

(c) What is the mean square fluctuation in n? That is, what is $<\Delta n^2> = <(n - <n>)^2>$?

Part 2

Suppose that an external magnetic field B_0 is now impressed on the system such that the spins are either aligned or anti-aligned with the field. If aligned the interaction energy is $-\mu B$, if anti-aligned $+\mu B$.

(d) Write an equation for the total energy of the system if n of the N spins are aligned with the field.

(e) Obtain the partition function of the system.

(f) Calculate the energy of the system as a function of temperature. Obtain expressions for the limiting forms of the energy at high and low temperatures. Sketch a graph showing the temperature dependence of the energy.

(g) Calculate the specific heat of the system as a function of temperature. Obtain expressions for the limiting forms of the specific heat at high and low temperatures. Sketch a graph showing the temperature dependence of the specific heat.

Part 3

Suppose that each spin interacts with its nearest neighbor such that there is a tendency for them to align with one another. That is suppose the interaction energy for adjacent spins is

$$\varepsilon = \begin{cases} -J & \text{(both spins up)} \\ -J & \text{(both spins down)} \\ +J & \text{(one spin up, one spin down)} \end{cases}$$

This can be summarized by writing for a neighboring pair of spins, say, i and j,

$$\varepsilon = -J\sigma_i\sigma_j$$

where $\sigma = +1$ if the spin is up and $\sigma = -1$ if the spin is down..

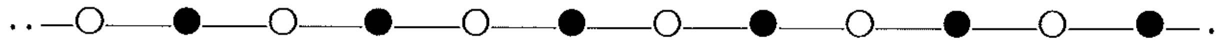
(h) Write an equation for the total energy of the N-particle system.

(i) Use this to write the partition function for the system as a sum over states.

(j) If the external magnetic field is zero, carry out the sum to obtain the partition function.

PhD/MS Comprehensive Examination
Fall 2004
Statistical Mechanics: SM600-2

Consider a one-dimensional crystal of length L made up of two types of atoms that alternate in the crystal. There are N atoms of each type.



Assume that at low temperatures the system can be treated as a system of Bose-type elementary excitations or quasi-particles. Two types of quasi-particles must be considered; the dispersion relations for them are

Type 1: $\omega = ck$

and

Type 2: $\omega = \omega_0$.

In these relations c and ω_0 are constants. Compute the low temperature contributions to the specific heat from each type of excitation.

The following may be helpful:

$$\int_0^\infty dx \frac{x}{e^x - 1} = \frac{\pi^2}{6} \qquad \text{and} \qquad \int_0^\infty dx \frac{x}{e^x + 1} = \frac{\pi^2}{12}$$

Consider a spinless particle in a uniform magnetic field. The Hamiltonian is given by

$$H = \frac{1}{2m}\vec{\Pi}^2 \quad \text{with} \quad \vec{\Pi} = \vec{p} - q\vec{A}\ , \quad \vec{B} = \nabla \times \vec{A}(\vec{r})\ , \quad \vec{B} = B\hat{z}\,.$$

(a) Find the commutators of $[\Pi_x, \Pi_y]$, $[\Pi_x, \Pi_z]$, $[\Pi_y, \Pi_z]$, and obtain a complete set of simultaneous base kets.

(b) Using ladder operators (similar to the harmonic oscillator problem), show that the energy eigenvalues are given by: $E = \frac{1}{2m}p_z^2 + (n + \frac{1}{2})\hbar\,\omega$. Determine ω.

(c) Consider $[H, \vec{r}]$ and show that $\tilde{X} = x + \frac{1}{qB}\Pi_y$ and $\tilde{Y} = y - \frac{1}{qB}\Pi_x$ are constants of the motion. Do they have simultaneous eigenvalues? If not, calculate the uncertainty relation for this pair of operators.

Consider an isotropic harmonic oscillator of mass m and angular frequency ω in **two** dimensions.

(a) What are the energies of the three lowest-lying states? Is there any degeneracy?

(b) Apply a perturbation $V = \delta\, m\omega^2 xy$, where δ is a very small dimensionless real number.

Find the lowest-order energy eigenket and the corresponding energy (to first order) for each of the three lowest-lying states.

(c) Solve the problem exactly and compare with the perturbation results obtained in *(b)*.

Hint: Substitute $P_\pm = \frac{1}{\sqrt{2}}(P_x \pm P_y)$; $X_\pm = \frac{1}{\sqrt{2}}(X \pm Y)$ in the Hamiltonian.

A spherical cavity of radius R is bounded by a thin wall of given opacity Ω defined by

$$V(r) = \frac{\hbar^2}{2m}\frac{\Omega}{R}\,\delta\,(r - R)$$

Investigate the s-wave scattering of a particle incident on this cavity.

(a) Write down the Schroedinger equation for the radial wave function $\chi_0(r)$ and consider the behavior of $\chi_0(r)$ and its derivative at the singularity of the potential in terms of the logarithmic derivative, $L_0(r) = r\,\dfrac{\chi_0'(r)}{\chi_0(r)}$.

(b) Use the boundary condition, resulting from the consideration in *(a)*, to determine the amplitude of $\chi_0(r)$ inside the cavity.

(c) Discuss the case of large opacity ($\Omega \gg 1$) and find energies at which resonant behavior occurs.