CUA

THE CATHOLIC UNIVERSITY OF AMERICA

Department of Physics
200 Hannan Hall
Washington, D.C. 20064
202-319-5315
Fax 202-319-4448

MS Comprehensive Examination
Physics Department

Fall 2003

Thursday, October 23, and Friday, October 24, 2003

Room 133 - Hannan Hall

GENERAL INSTRUCTIONS:

This examination is divided into four sections as follows:

<u>Thursday, October 23, 2003</u>

9:00 a.m. - 12:00 Noon	Classical Mechanics - 2 questions
1:00 P.M. - 5:00 P.M.	E & M - 2 questions

<u>Friday, October 24, 2003</u>

9:00 a.m. - 12:00 Noon	Thermodynamics/Stat. Mech. - 2 questions
1:00 P.M. - 5:00 P.M.	Modern Physics/Quantum Mech. - 2 questions

<u>DO EACH PROBLEM IN A SEPARATE BLUE BOOK</u>

PUT YOUR NAME ON EACH BOOK

LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS
FOR EXAMPLE: Mechanics 600-1 or Electromagnetism 500-2

YOU <u>MUST</u> DO TWO (AND ONLY TWO) OF THE QUESTIONS IN EACH AREA.

OF THE TOTAL OF EIGHT QUESTIONS THAT ARE DONE, A <u>MINIMUM</u> OF TWO MUST BE FROM THE 600-LEVEL QUESTIONS.

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RULES FOR THE PHYSICS COMPREHENSIVE EXAMINATION

Fall 2003

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1. The examination will be closed book. Students will not be permitted to bring to the examination room materials such as notes, reference books, calculators, or other aids of any form. The Physics Department will supply copies of the *CRC Mathematical Handbook,* Schaum's *Mathematical Handbook, Table of Functions* by Jahnke and Emde, and the NBS *Handbook of Mathematical* Functions, for the examinees' use should they feel the need for these references during the examination.

   Students who want to use their own mathematical tables and/or handbooks must get permission from the Chairman at least two days before the beginning of the comprehensive. These tables and/or handbooks will remain under the control of the department until after the comp exam but will be available to their owners during the exam.

2. If needed, students will be allowed a short break, not to exceed 5 minutes, to use the restroom facilities during the examination period. While absent from the examination room students may not consult material or other sources of information concerning any matter pertaining to the examination. Students who leave the examination room will turn in all of their examination materials to the proctor before leaving the room. Examination papers will be returned to the students when they reenter the examination room. Students who fail to return within 5 minutes will be considered to have completed that portion of the examination.

Only one student will be permitted to leave the examination room at a time.

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THE CATHOLIC UNIVERSITY OF AMERICA

Preliminary Examination--Physics Dept.

Thursday, October 23, and Friday, October 24, 2003

Room 133 - Hannan Hall

- **YOU MUST DO TWO QUESTIONS IN EACH OF THE AREAS:**

Thursday, October 23, 2003

Mechanics

Electricity & Magnetism

Friday, October 24, 2003

Thermodynamics

Modern Physics/Quantum Mechanics

- **DO EACH PROBLEM IN A SEPARATE BLUE BOOK**
- **PUT YOUR NAME ON EACH BLUE BOOK**
- **LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS**
  FOR EXAMPLE: Mechanics #1

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Ph.D. Comprehensive Examination

Physics Department

Fall 2003

Thursday, October 23, and Friday, October 24, 2003

Room 133 - Hannan Hall

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1:00 P.M. - 5:00 P.M.	E & M - 3 questions

<u>Friday, Octobwe 24, 2003</u>

9:00 a.m. - 12:00 Noon	Stat Mech. - 2 questions
1:00 P.M. - 5:00 P.M.	Quantum Mech. - 3 questions

DO EACH PROBLEM IN A SEPARATE BLUE BOOK

PUT YOUR NAME ON EACH BOOK

LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS
FOR EXAMPLE: Mechanics 600-1

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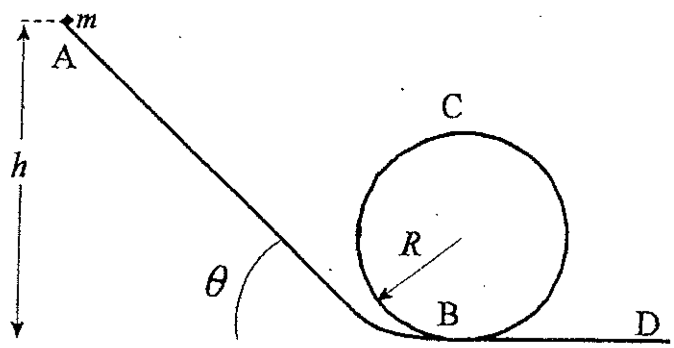
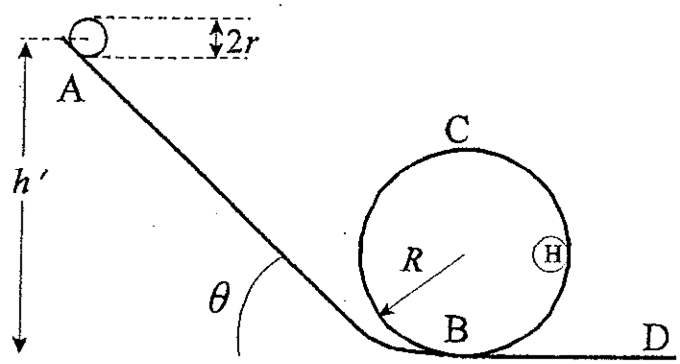
(a) A point particle of mass m starts at point A and slides down a frictionless ramp inclined at an angle θ from the horizontal. It starts at an initial height h as shown in the sketch below and moves around the frictionless loop of radius R along the path ABCBD.



What is the minimum height h such that the mass will stay on the track throughout its motion?

(b) Now suppose that, instead of being a point mass, the object is a small cylinder of radius $r \ll R$, mass m and moment of inertia I. It starts at rest at a new height h' and rolls without slipping down the track, around the circular loop, and continues rolling to point D.



(i) What is the minimum starting height h that is needed if the cylinder is to roll around the loop without falling off? You may assume that the cylinder's height at the top of the loop (at point C) is approximately $2R$ since $R \gg r$.

(ii) What is the normal force that the track exerts on the cylinder when it is at the point H halfway to the top of the loop?

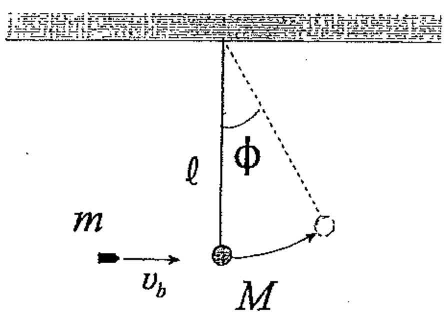
Express your answers in terms of R, r, m, I, θ, and g (the acceleration due to gravity) only.

MS Comprehensive/Preliminary Examination
Fall 2003
Mechanics 500-2

A simple way to measure the speed of a bullet is with a *ballistic pendulum*. As illustrated, we have a mass, M, suspended on a mass-less rod of length l. The bullet is shot and embeds itself into the mass, causing the rod to swing through a maximum angle φ. Assume that the bullet and the mass are point masses and the rod can only move in two dimensions: horizontally (in the direction of the bullet) and vertically. The initial speed of the bullet is v_b and its mass is m.

a) How fast is the target mass moving immediately after the bullet comes to rest in it? Assume this happens very quickly. Answer in terms of the given quantities.
b) Determine the maximum displacement angle φ in terms of the given quantities.
c) Now, let's consider the case when the bullet/target interaction is *purely elastic* (e.g., it's a rubber bullet, of the type certain security forces are fond of using). Determine the maximum displacement angle φ for this situation.

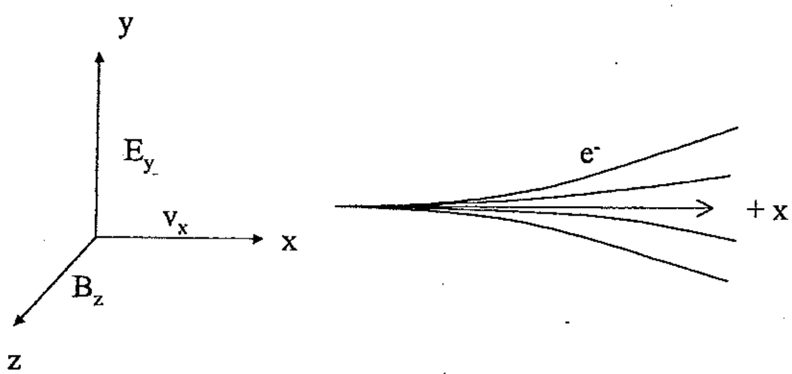
Hint: Remember what quantities are conserved in both these cases.



Consider a narrow beam of electrons initially moving along the +x axis with an initial displacement, $x = x_0$, $y = y_0$ at time, $t = 0$. There is a uniform, constant electric field directed perpendicular to its motion along the +y axis. Initially, the beam of electrons has a small spread of velocities around an average velocity, v_0, directed along positive x. There is also a uniform, constant magnetic field present, which is directed along the +z axis.

a.) Suppose that the strength of the uniform magnetic field can be varied. Show through a derived expression that there is some distinct value of **B**, where the electrons having the initial velocity v_0 along x and passing through the region common to both **B** and **E** will be undeviated by the forces produced by **E** and **B**.

b.) If the **B** - field is adjusted to the situation described in b.), show for small angular deflections that electrons in this beam, for any other velocity around v_0, will follow a trajectory $y(x) = y_0 + k(x - x_0)^2$. Here, k is a constant ultimately depending only upon v, over the small initial velocity range along the x-axis. You are to determine the dependence of k on v.
(Assume that $v_x \gg v_y$ such that we have small angular deflections for the electrons in the region containing uniform **E** and **B**.)

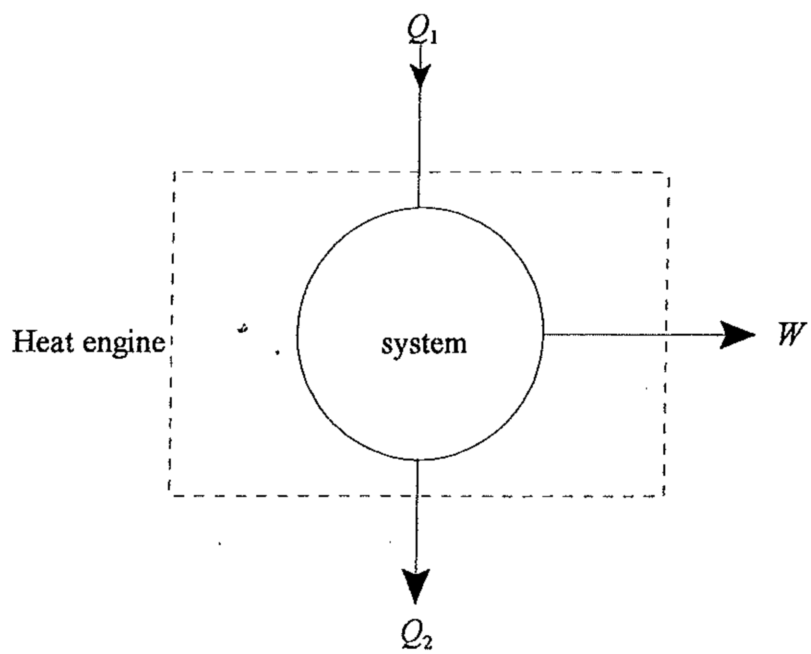


A sphere of uniform, fixed positive charge is present in a vacuum. The sphere has a charge density ρ and a radius R.

a) Find the electric field at a distance r ($r < R$) from the center of the sphere of charge.
b) Find the electric field at a distance r ($r > R$) from the center of the sphere.
c) Determine how much work is done bringing a positive charge Q from infinity to the center of the sphere, assuming that the dielectric constant of the sphere is ≈ 1.
d) If the sphere is made of a dielectric material whose dielectric constant is $k > 1$, and the same charge is uniformly embedded in the sphere, determine the change in the work necessary to bring the charge Q in from infinity (the change, that is, compared to the work you calculated in part c.). Note: outside the sphere it is still a vacuum.
e) Give a physical explanation for the difference in the work found in part d. compared to part c.

There is one particularly simple cyclic process which plays an important part in the development of thermodynamics. Known as *Carnot cycle*, it consists of four distinct processes:

(a) The working substance expands isothermally and reversibly at temperature T_1 absorbing heat Q_1.



Application of the first law to a heat engine.

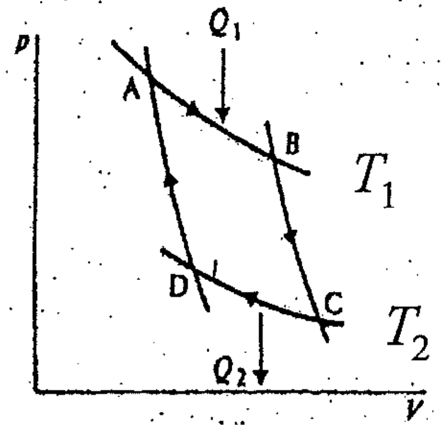
(b) The working substance expands adiabatically and reversibly, the temperature changing from T_1 to T_2.

(c) The working substance is compressed isothermally and reversibly at T_2 rejecting heat Q_2.

(d) The working substance is compressed adiabatically and reversibly from T_2 to the initial state T_1.

The Carnot cycle for a simple system is illustrated below.

Carnot cycles in a ideal gas. AB and CD are isothermal processes. BC and DA are adiabatic.



1) For an ideal gas, calculate the work done and the heat used per cycle. Then calculate the efficiency. Assume the higher temperature is T_1 and the lower temperature is T_2.
2) For the same assumptions, calculate the entropy increase per cycle.

Derive these two useful *"TdS* Equations":

$$TdS = C_V dT + \frac{\alpha T}{\kappa_T} dV \quad , \qquad (1)$$

$$TdS = C_P dT - \alpha V T dP \quad . \qquad (2)$$

Here, the Thermodynamic System involves a single species with a constant number N of moles (or particles). Alternatively, you may regard S, V, C_V, C_P as molar quantities, and use correspondingly molar equations throughout. Furthermore,

$C_V = \left(\frac{\delta Q}{\delta T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V$ and $C_P = \left(\frac{\delta Q}{\delta T}\right)_P = T\left(\frac{\partial S}{\partial T}\right)_P$ are Heat Capacities (or specific) at constant Volume and Pressure, respectively. Also, $\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P$ and $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$ are the coefficients of Thermal Expansion and Isothermal Compressibility, respectively.

<u>Hint</u>: A possible approach is to start from $S = S\,(U,V)$ and its exact differential

$dS = \left(\frac{\partial S}{\partial U}\right)_V dU + \left(\frac{\partial S}{\partial V}\right)_U dV = \frac{1}{T} dU + \frac{P}{T} dV$. Then consider, for Eq. (1), $S = S\,(T,V)$ and its exact differential. The (T,V) dependence suggests to handle the partial derivatives of S through a Legendre Transformation $U[T] = U - TS = F\,(T,V)$ and the corresponding Maxwell Relation for the second partial derivatives of $F\,(T,V)$. For Eq. (2), consider $S = S\,(T,P)$ and its exact differential. The (T,P) dependence suggests the Legendre Transformation $U[T,P] = U - TS + PV = G\,(T,P)$, etc.

MS/ PhD Comprehensive Examination
Fall 2003
Quantum Mechanics 500-1

Make use of the uncertainty principle to answer the following.

(a) (20 points) If a certain excited state of a nucleus is known to have a lifetime of 5.0×10^{-14} s, what is the minimum accuracy within which its energy can be measured.

(b) (30 points) An electron with kinetic energy 25 keV traveling in the $+x$ direction in free space is localized within a region of space $\Delta x = 0.50$ nm. Its wavefunction can be written as

$$\psi(x,t) = \int_{-\infty}^{+\infty} dk\, \varphi(k)\, e^{i(kx - Et/\hbar)}$$

Estimate the spread of wavevectors Δk over which the function $\varphi(k)$ differs appreciably from zero. Is this a sufficiently narrow spread so that one may meaningfully speak of a momentum for the electron? Explain.

(c) (50 points) Consider a hydrogen atom and let r designate the distance from the nucleus to the electron. Designate the uncertainty in the position of the electron in the ground state as $\Delta r \equiv a$.

(i) Use the uncertainty principle in the form $\Delta p_r\, \Delta r \approx \hbar$ to write the momentum uncertainty in the ground state.

(ii) Use this to estimate the ground state kinetic energy.

(iii) Then take $-\,e^2/4\pi\varepsilon_0 a$ as an estimate of the potential energy and write an expression for the total energy of the hydrogen atom in terms of e, $\hbar$, ε_0, a and the electron mass m.

(iv) Obtain an expression for the value of a (call it a_0) that minimizes the energy and then use this value to write down the minimum energy E_0.

(v) Finally evaluate a and E_0 numerically.

Take $\hbar = 6.59 \times 10^{-16}$ eV·s $= 1.054 \times 10^{-34}$ J·s and $\varepsilon_0 = 8.854 \times 10^{-12}\ C^2/N\text{-}m^2$. For an electron $m = 9.11 \times 10^{-31}$ kg $= 0.511$ MeV/c^2 and $e = 1.60 \times 10^{-19}$ C.

MS Comprehensive/Preliminary Examination
Fall 2003
Quantum Physics 500-2

Consider a step-function, V(x), where $V(x) = V_0$ for $x > 0$ and $V(x) = 0$ for $x < 0$. Let the energy of a particle moving from the $-x$ direction has an energy $E < V_0$.

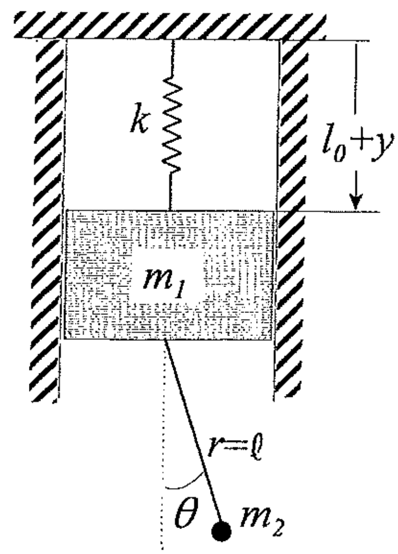
a.) What are the general forms of the solution to the time-independent Schroedinger Equation in the domains of $x < 0$ and $x > 0$?

b.) Show that the general forms of the solution is, indeed, a solution of the the time-independent Schroedinger Equation.

c.) Derive the probabilities for reflection and transmission for the particle at x in the neighborhood of $x = 0$. Specifically, show quantum mechanically that the probability of reflection = 1 for $E < V_0$.

d.) Repeat the steps leading to get the results in c.), except now assume the energy of the particle is $E > V_0$. Now, what is the probability of reflection and transmission?

In the arrangement shown at the right, m_1 is suspended from a spring of spring constant k and unstretched length l_0. This mass is allowed vertical motion only. The particle m_2 is confined to move in a plane with $r = l =$ constant.



You may assume that the θ-motion is such that m_2 does not strike the walls that constrain m_1's motion. In parts *(a)* and *(b)* do ***not*** assume that θ is small.

(a) Obtain the Lagrangian for arbitrary values of the angular coordinate θ.

(b) Obtain Lagrange's equations of motion for the generalized coordinates y and θ.

(c) Simplify these by assuming that θ is restricted to small values (do ***not*** assume that y is necessarily small). Linearize the equations by neglecting terms of second order smallness.

(d) What is the frequency of oscillation of the mass m_1?

Consider the motion of a particle in a central inverse-square-law force field for the case in which there is a superimposed force whose magnitude is inversely proportional to the cube of the distance from the particle to the force center. That is:

$$F(r) = -\frac{\kappa}{r^2} - \frac{\lambda}{r^3}, \text{ where } \kappa, \lambda > 0.$$

Specifically, show that the motion is **described by a precessing ellipse**. Do your derivation for the case in which $\lambda << l^2/\mu$, where l is angular momentum and μ is the reduced mass.

To get started, find the equation of motion, then use the following substitutions:

$u = 1/r$, $du/d\theta = -(\mu\, dr/dt)/l$ (don't get u and μ confused!)

Once you have the equation of motion, try a solution for the Newtonian terms in the form:

$u_1 = const. \times (1 + \varepsilon \cos\theta)$, noting that ε is the eccentricity.

Other hints: you will need to add a term to the Newtonian solution, e.g., u_p, to handle the extra force. This should include a term proportional to θ. Once you get your solution, isolate the secular terms and show (using the small angle approximations for sine and cosine) that it can be written as $(1 + \varepsilon \cos(\theta - \frac{\mu\lambda\theta}{l^2}))$.

MS/ PhD Comprehensive Examination
Fall 2003
Electricity and Magnetism 600-2

Consider a spherical shell of radius R with a fixed charge distribution of charge surface density $\sigma = \sigma_0\cos(\theta)$, where σ_0 is a constant and theta is the polar angle. Interior and exterior to this shell is a vacuum with no charges present.

a.) Calculate the potential inside and outside the shell.

b.) Calculate the electric field, $\vec{E}$, interior and exterior to the shell.

A linearly polarized light wave moving in the $+z$ direction in free space has a finite extent in the x- and y-directions, described by a gaussian amplitude function

$$E_0(x,y) = E_{00}\, e^{-(x^2+y^2)/4\sigma^2}.$$

Assume that the amplitude modulation is slowly varying, that is, the beam is many wavelengths broad, *i.e.*, $k\sigma \gg 1$.

(a) Show that although

$$\mathbf{E}(x,y,z) = \hat{\mathbf{i}}\, E_0(x,y)\, e^{i(kz-\omega t)},$$

where $\hat{\mathbf{i}}$ is a unit vector in the x direction, is an approximate solution to the wave equation correct up to terms of order $1/k^2\sigma^2$, it is **not** an acceptable description of the field.

(b) Show that

$$\mathbf{E}(x,y,z) \approx \left[E_0(x,y)\,\hat{\mathbf{i}} + \frac{i}{k}\left(\frac{\partial E_0}{\partial x}\right)\hat{\mathbf{k}}\right] e^{i(kz-\omega t)}.$$

gives an approximately acceptable expression for the electric field. Write this out explicitly for the specific form of $E_0(x,y)$ given above. Here $\hat{\mathbf{k}}$ is a unit vector in the z direction.

(c) Note that, although the wave is propagating in the $+z$ direction, the field has a component in this direction. How is one to interpret this apparent longitudinal component of the wave?

HINT: The field must satisfy Maxwell's equations.

The object of this question is to calculate the translational, rotational and vibrational contributions to the specific heat of diatomic nitrogen gas for elevated temperatures, say temperatures above 300K. Specifically, consider a dilute gas of N molecules of N_2 confined to a volume V at temperature T.

Begin by arguing that the translational and rotational degrees of freedom can be treated classically, but that the vibrational degree of freedom must be analyzed using quantum mechanics. Then calculate as a function of temperature

(a) the contribution to the specific heat arising from the translational degrees of freedom of the N_2 molecule;

(b) the contribution to the specific heat arising from the rotational degrees of freedom of the N_2 molecule; and

(c) the contribution to the specific heat arising from the vibrational degree of freedom of the N_2 molecule.

Data: For nitrogen

$$\Theta_{rot} = \frac{\hbar^2}{2Ik_B} = 2.847\ \text{K}$$

where I is the moment of inertia of the nitrogen molecule about an axis through its center of mass and perpendicular to the molecule's symmetry axis, and k_B is Boltzmann's constant.

$$\Theta_{vib} = \frac{\hbar\omega}{k_B} = 918.9\ \text{K}$$

where ω is the molecule's angular vibration frequency.

Consider a ultra-relativistic ideal Fermi gas in 2-dimensions. For each fermion, the energy-momentum relation then is $\varepsilon = c\sqrt{p_x^2 + p_y^2}$. Suppose that there are N fermions confined to an area A.

(a) Determine the Fermi Momentum p_F, and from that the Fermi Energy ε_F.
Hint: what is the area of the "Fermi ball" in 2-dimensions?

(b) Determine the density of states $g(\varepsilon)\,d\varepsilon$ of the ultra-relativistic 2-dimensional ideal Fermi gas. Include a spin-degeneracy factor $g_s = 2$.

(c) Verify that $N = \int_0^{\varepsilon_F} g(\varepsilon)d\varepsilon$ from (a) and (b).

(d) Assume that the gas is kept at a temperature $T \ll \varepsilon_F / k_B$.
Determine the Internal Energy $U_o = \int_0^{\varepsilon_F} \varepsilon\, g(\varepsilon)d\varepsilon$ in terms of N and ε_F.

(e) Show that $PA = \frac{1}{2}U$ holds in fact at any temperature T. Here the pressure P denotes the force per unit length exerted on the perimeter of the A area confining the gas.

Hint: recall that $\frac{PA}{k_B T} = \log \mathcal{L} = \int_0^\infty \log(1 + z e^{-\beta\varepsilon})\, g(\varepsilon)d\varepsilon$ and $U = -\frac{\partial}{\partial\beta}(\log \mathcal{L})_{z,A}$. Change variables of integration as $x = \beta\varepsilon$, in order to extract the relevant β- dependence.

Consider a particle of mass M confined inside an infinite cylinder of radius a, such that $V(\rho,z,\varphi) = 0$ for $\rho < a$ and $V(\rho,z,\varphi) = +\infty$ for $\rho > a$. In cylindrical coordinates, the Schroedinger Equation (S.E.) becomes

$$-\frac{\hbar^2\nabla^2}{2M}\psi + V\psi = -\frac{\hbar^2}{2M}\left\{\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}\right\}\psi(\rho,z,\varphi) + V(\rho)\psi(\rho,z,\varphi) = E\psi(\rho,z,\varphi).$$

Proceed to solve the $S.E.$ by separation of variables, setting

$$\psi(\rho,z,\varphi) = R(\rho)Z(z)Q(\varphi),\; E = E_\perp + E_\parallel = E_\perp + \frac{\hbar^2 k_z^2}{2M}.$$

(a) Determine the differential equations for $Z(z)$, $Q(\varphi)$ *and* $R(\rho)$, and the corresponding separation constants.

(b) Make a change of variables $x = \frac{\sqrt{2ME_\perp}}{\hbar}\rho$ in the differential equation for $R(\rho)$, thus reducing it to <u>Bessel Equation</u>

$$\left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + \left(1 - \frac{m^2}{x^2}\right)\right]J_m(x) = 0.$$

(c) By enforcing the boundary condition $R(\rho = a) = 0$, determine the possible discrete values of $E_\perp$, which represents the energy contribution of the motion in the horizontal plane. In particular, what is the Ground State Energy?

(d) In the Laplacian ∇^2, identify the L_z^2 contribution. In cylindrical coordinates, what is the expression of the z-component of the angular momentum operator L_z ?

(e) Which of the following operators represent constants of the motion, and why:

$P_x, P_y, P_z, L_x, L_y, L_z, L^2$ and z-reflection?

MS/ PhD Comprehensive Examination
Fall 2003
Quantum Mechanics 600-2

A particle of mass m is confined in a one-dimensional infinite square-well potential $V(x)$ with $V(x) = 0$ for $0 \leq x \leq a$ and $V(x) = +\infty$ otherwise.

(a) Determine the normalized eigenfunctions $\varphi_n(x)$ and the energy eigenvalues E_n of the Hamiltonian $H = \frac{P^2}{2m} + V(x)$, for $n = 1,2,3,...$

(b) At $t = 0$, the particle is in a superposition wavefunction

$$\psi(x,o) = \frac{1}{\sqrt{2}}\varphi_1(x) - \frac{i}{\sqrt{2}}\varphi_2(x).$$

Determine the average momentum at $t = 0$,

i.e., $<P>_{t=0} = <\psi(x,0)\,|P|\,\psi(x,0)>$, where $P = -i\hbar\frac{\partial}{\partial x}$ is the momentum operator.

Useful Formula: $\sin\alpha\cos\beta = \frac{1}{2}\{\sin(\alpha+\beta) + \sin(\alpha-\beta)\}$.

(c) Recalling the time-evolution $\psi(x,t)$ of $\psi(x,o)$, determine the average momentum of the particle as a function of time t, i.e., $<P>_t = <\psi(x,t)\,|P|\,\psi(x,t)>$. What is the frequency of the oscillation of $<P>_t$? Is P a "constant of motion", and, if not, why not?

MS/ PhD Comprehensive Examination
Fall 2003
Quantum Mechanics 600-3

The one dimensional harmonic oscillator problem can be solved by defining ladder operators

$$\hat{a} \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \quad \text{and} \quad \hat{a}^{+} \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$$

where $\hat{x}$ and $\hat{p}$ are the position and momentum operators.

a.)Using the commutation relation $[\hat{x},\hat{p}] = i\hbar$, show that $\hat{a}$ and $\hat{a}^{+}$ obey the commutation relation $[\hat{a},\hat{a}^{+}] = 1$.

b.) Show the Hamiltonian can be written in the form $\hat{H} = \hbar\omega(\hat{a}^{+}\hat{a} + \frac{1}{2})$.

c.) Consider the operator $\hat{N} \equiv \hat{a}^{+}.\hat{a}$. Using only the commutation relation, show that if Ψ_n is an eigenfunction of $\hat{N}$ with eigenvalue n, then $\hat{a}^{+}\Psi_n$ is an eigenfunction of $\hat{N}$ with eigenvalue n + 1.

d.) Show how the result derived in c.) leads to the correct spacing of the energy eigenvalues of the harmonic oscillators.

e.) Now consider the ground state $|0\rangle$ and its annihilation by the $\hat{a}$-operator as $\hat{a}|0\rangle = 0$. In the x-coordinate representation, write the latter as a first-order differential equation for the ground-state wave-function $\Psi_0(x) = \langle x|0\rangle$.

<u>*Hint*</u>: $\langle x|\hat{p}|\psi\rangle = -i\hbar\frac{\partial}{\partial x}\Psi(x)$.

f.) Solve $\langle x|\hat{a}|0\rangle = 0$ and determine $\Psi_0(x)$.

g.) Verify $\Psi_0(x)$ satisfies the Schroedinger equation for the harmonic oscillator with $E_0 = \frac{1}{2}\hbar\omega$.

h.) With $\Psi_0(x)$ determined, how can the eigenfunction $\Psi_n = \langle x|n\rangle$ also be derived in a recursive manner for n = 1, 2, 3, ...?

<u>*Hint*</u>: Consider the coordinate representation of $\langle x|\hat{a}^{+}|\psi\rangle$.