A billiard ball with radius \( R \) and mass \( M \) rests on a flat, horizontal table. The ball is struck along its horizontal axis (as shown) so that it moves forward with an initial velocity \( v_0 \). As the ball moves it experiences sliding friction with a coefficient \( \mu \). The friction causes a torque, which increases the rotation rate of the ball until it rolls without slipping.

(a) Draw a diagram showing the forces and torques acting on the ball.
(b) Write down the equations of motion of the ball before it stops slipping.
(c) How long does it take until the ball rolls without slipping?
(d) What is the translational velocity at the time the ball ceases to slip?

HINT: The moment of inertia of a sphere is \( \frac{2}{5} MR^2 \).

A particle of mass \( m \) moves under the influence of an attractive central force \( f(r) \).

(a) Show that by a proper choice of the initial velocity a circular orbit of radius \( r_0 \) a circular orbit will result.
(b) The circular orbit is subjected to a small radial perturbation. Determine the relationship that must hold between \( f(r) \) and \( df/dr \) for the orbit to be stable.
(c) If the force law is of the form \( f(r) = -\frac{K}{r^n} \) (with both \( K \) and \( n \) positive constants), determine the maximum value of \( n \) for which a circular orbit is stable.

HINT: It may help to make a sketch of the effective potential \( U(r) \) versus \( r \). To obtain the effective potential try writing the angular momentum of the system and then substitute this in the equation for the total energy of the system.
A 100-g sample of a pure substance is heated to a temperature of 50°C where it is a liquid. It is then removed from the furnace and allowed to cool at constant pressure to room temperature (20°C). Assume that Newton’s law of cooling for the rate of heat loss
\[ \frac{dT}{dt} = -k(T - T_{\infty}) \]
applies. Here \( T_{\infty} \) is the temperature of the environment.

The temperature is measured at two-minute intervals after the sample is removed from the furnace and the data at the right are obtained. From these data and the known specific heat of the substance in solid form (0.050 cal/g°C) and the latent heat of fusion, determine:
(a) the melting point of the substance;
(b) the latent heat of fusion;
(c) the entropy change associated with melting.

State explicitly any additional assumptions that you make.

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The temperature of Earth's surface is determined by the balance between incoming solar radiation and outgoing infrared radiation (see diagram at right). The solar intensity is
\[ S = 1.4 \text{ kW/m}^2 \]
and the infrared intensity is
\[ I = 0.15 \text{ W/m}^2 \text{·K} \]
where \( k = 5.7 \times 10^{-8} \text{ W/m·K} \).

(a) By assuming that Earth does not heat up on the average, write down the equation of energy conservation between the incoming and outgoing radiation. Allow for the fact that 30% of the solar radiation is reflected back out to space without being absorbed. Assume that the infrared emission of Earth's surface is characterized by an average temperature \( T_{\text{Shell}} \).

(b) Insert what you know and solve for \( T_{\text{Shell}} \).

A more realistic model takes into account the greenhouse warming effect of the atmosphere. This is done by treating the atmosphere as a spherical shell enclosing Earth. Assume that the shell has the same area as Earth's surface. It is transparent to the incoming solar radiation but absorbs some of the outgoing infrared radiation from Earth's surface. It is thus heated and radiates its own infrared radiation, from both its upper and lower surfaces, with intensity \( I = 0.15 T_{\text{Shell}}^{4} \) (see diagram).

(c) Write the equation of energy conservation for this shell. Let the temperature of Earth be \( T_{E} \) and that of the shell be \( T_{\text{Shell}} \). Assume that the shell absorbs a fraction \( f \) of the outgoing infrared radiation, and that its surface area (outer and outer) is the same as Earth's.

(d) Use this to express the temperature of the shell \( T_{\text{Shell}} \) in terms of Earth's surface temperature \( T_{E} \).

(e) Write a new equation of energy conservation for Earth's surface which includes the downward infrared radiation from the shell, which is absorbed by Earth's surface.

(f) Finally, calculate a new value for the temperature of Earth's surface \( T_{E} \). Use \( f = 0.75 \). This new value includes the greenhouse warming.
The moment of inertia tensor can be expressed by

\[ I = \int \rho \left( \sum_{i=1}^{3} \delta_i \sum_{j=1}^{3} x_i x_j \right) \, dV, \]

where \( \rho \) is the density and \( i, j \), and \( k \), are the indices for the three axes.

(a) Calculate the moment of inertia tensor for a cube of dimension \( a \) for the case where the three axes lie along the edges of the cube.

(b) Use this tensor to calculate the energy of rotation for a cube rotating with an angular velocity \( \omega \) about an axis given by \( x = y \).

Two pendulums with identical lengths \( l \) and masses \( m \) are connected by a linear spring of spring constant \( A \) (see the sketch at the right). Their motion is confined to the \( xz \) plane (the plane of the paper).

(a) Write the Lagrangian \( L \) for this system.

(b) Find the equations of motion for the system using

\[ \frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial}{\partial \theta_i} \frac{\partial L}{\partial \dot{\theta}_i} = 0 \quad (i = 1, 2). \]

(c) Taking the limit of small displacements, find the normal modes and associated frequencies of this two pendulum system. Describe the normal modes.

(d) The system is started at \( t = 0 \) with \( \dot{\theta}_1(0) = \dot{\theta}_2(0) = 0 \), \( \ddot{\theta}_1(0) = \ddot{\theta}_2(0) = 0 \) and \( \dot{\theta}_3(0) = 3 \text{ rad/s} \). Find its subsequent time evolution.
Consider a dilute gas of $N$ molecules confined to a chamber of volume $V$ where it is in equilibrium at some high temperature $T$. Through a window in the chamber containing the gas, one observes a spectral line characteristic of some electronic transition. Because of the molecules' thermal motion the observed spectral line is Doppler broadened.

(a) Show that the intensity function describing the shape of the line as a function of frequency $\nu(x)$ is a Gaussian function centered at $\omega_0$, the spectral frequency for a molecule at rest. Treat the translational degrees of freedom classically.

HINT: Recall that the Doppler shifted frequency $\nu$ for a light source moving toward $\nu_1$ or away from an observer $\nu_2$ at a speed $v \ll c$ is

$$\omega = \frac{\nu_0}{1 \pm \nu v/c}$$

where $\nu_0$ is the frequency observed in the rest frame of the source.

(b) Obtain the temperature dependence of the width of the line.

Suppose that the gas is composed of rigid symmetric-top molecules (two equal moments of inertia). The rotational energy levels are given by

$$\varepsilon_{j, \lambda, \lambda'} = \frac{\hbar^2}{2I_1} \left( \frac{\lambda^2 + 1}{\lambda^2} \right) \left( \frac{1}{I_1} - \frac{1}{I_2} \right) = \hbar^2 \left( j + 1 \right) + \gamma \lambda^2$$

where $I_1$, $I_2$, and $I_3$ are the principal moments of inertia and where the quantum numbers $j$, $\lambda$, and $m$ take the values

$$0 \leq j = m = -j \leq \lambda \leq j.$$  

The energy is degenerate with respect to $m$. Assume $I_1 > I_2$ so that $\gamma > 0$.

(c) Continuing to treat the translational motions classically, obtain the total partition function of the system in the form

$$Z(V, T) = \sum_{j, \lambda, \lambda'} \sum_{\beta} \varepsilon_{j, \lambda, \lambda'}^{\beta} \left( \beta \lambda \gamma \right)^{\beta} \beta^\lambda \lambda$$

where $\beta = 1/kT$ and $f$ and $g$ are functions that you must determine. Here $f(N, V, T)$ is the translational partition function and the other factor is the rotational partition function.

(d) From this obtain the specific heat of the gas in the limit of high temperatures.

HINT: For part (d) use the Bolz-MacLaurin formula

$$\sum_{j, \lambda, \lambda'} \varepsilon_{j, \lambda, \lambda'}^{\beta} \beta^\lambda \lambda = \sum_{j, \lambda, \lambda'} \left( \frac{\hbar^2}{2I_1} \left( \lambda^2 + 1 \right) \left( \frac{1}{I_1} - \frac{1}{I_2} \right) \right) \left( \beta \lambda \gamma \right)^{\beta} \beta^\lambda \lambda,$$

where $f^{(m)}(n)$ denotes the $m$th derivative of $f$ and the $B_k$ are the Bernoulli numbers. Retain only the first term on the right and try integrating by parts.
Tritium, $^1\text{H}$, a nucleus with one proton and two neutrons, decays via beta emission to helium-3, $^3\text{He}$, a nucleus with two protons and one neutron. If the electron bound to the tritium nucleus is in the ground state before the decay, what is the probability that the electron will be in the ground state of the $^3\text{He}$ atom after the decay?

The radial wave function for the ground state of a one-electron atom has the form $e^{-Zr/\hbar}$, where $Z$ is the atomic number, $r$ is the radial coordinate, and $\hbar$ is the Bohr radius. Ignore the mass difference between the helium-3 and tritium nuclei.

Mathematical hint: $\int_0^{\infty} x^\alpha e^{-x} \, dx = \frac{\pi^{\frac{\alpha}{2}}}{\Gamma\left(\frac{\alpha}{2}\right)}$

Consider a particle of mass $m$ confined to a spherical well of radius $a$.

The potential $V(r)$ is:

$$V(r) = \begin{cases} 0 & r < a \\ -\infty & r \geq a \end{cases}$$

This might describe a particle confined in a spherical quantum dot.

(a) Begin with the time-dependent Schrodinger equation and separate out the time dependence to get the time-independent Schrodinger equation for the stationary states of the system.

(b) Again use separation of variables to derive separate differential equations for the radial and angular dependences of the eigenfunction. Show that the radial equation is:

$$\frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} + \left(\frac{\ell(\ell + 1)}{r^2} - \frac{\hbar^2}{2m} \right) R(r) = 0$$

where $R(r)$ is the radial function. Identify the constant $\ell^2$.

(c) The solutions to this are the spherical Bessel functions $j_\ell(\ell r)$ and the spherical Neumann functions $n_\ell(\ell r)$, the first few of which are given below. One of these must be discarded on physical grounds. Which one? Explain.

(d) Obtain the condition that determines the energy eigenvalues and use it to obtain an equation for the energies. Use this to calculate the ground state energy.

The first few spherical Bessel and Neumann functions are:

$$j_0(x) = \frac{\sin x}{x}, \quad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}; \quad j_2(x) = \left(\frac{3}{x^2} - \frac{1}{x}\right) \sin x - \frac{3 \cos x}{x^2}$$

and

$$n_0(x) = \frac{\cos x}{x}, \quad n_1(x) = \frac{\cos x}{x^2} - \frac{\sin x}{x}; \quad n_2(x) = \left(\frac{3}{x^2} - \frac{1}{x}\right) \cos x - \frac{3 \sin x}{x^2}$$
Under what conditions will the WKS create levels of the bond energy? The WKS approximation is such that

\[
\begin{align*}
& (\sigma < \epsilon) \\
& (\sigma \geq \epsilon) \\
& (\sigma / |\epsilon|) A^\pm - \}
& = (\sigma A

A particle of mass \( m \) moves under the influence of the one-dimensional potential.
Ph.D. Comprehensive Examination

Physics Department

Fall 2000

Thursday, October 26 and Friday, October 27, 2000

Room 133 - Hannan Hall

GENERAL INSTRUCTIONS:

This examination is divided into four sections as follows:

**Thursday, October 26, 2000**
9:00 a.m. - 12:00 Noon  Classical Mechanics - 2 questions
1:00 P.M. - 5:00 P.M.  E & M - 3 questions

**Friday, October 27, 2000**
9:00 a.m. - 12:00 Noon  Stat Mech. - 2 questions
1:00 P.M. - 5:00 P.M.  Quantum Mech. - 3 questions

**DO EACH PROBLEM IN A SEPARATE BLUE BOOK**

**PUT YOUR NAME ON EACH BOOK**

**LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS**
FOR EXAMPLE: Mechanics 600-1
THE CATHOLIC UNIVERSITY OF AMERICA
Department of Physics
200 Hanman Hall
Washington, D.C. 20064
202-319-5315
Fax 202-319-4448

MS Comprehensive Examination
Physics Department

Fall 2000

Thursday, October 26, and Friday, October 27, 2000

Room 133 - Hanman Hall

GENERAL INSTRUCTIONS:

This examination is divided into four sections as follows:

Thursday, October 26, 2000
9:00 a.m. - 12:00 Noon
Classical Mechanics - 2 questions

1:00 P.M. - 5:00 P.M.
E & M - 2 questions

Friday, October 27, 2000
9:00 a.m. - 12:00 Noon

1:00 P.M. - 5:00 P.M.
Modern Physics/Quantum Mech. - 2 questions

DO EACH PROBLEM IN A SEPARATE BLUE BOOK

PUT YOUR NAME ON EACH BOOK

LABEL EACH BOOK WITH CORRECT PROBLEM NUMBERS
FOR EXAMPLE: Mechanics 600-1 or Electromagnetism 500-2

YOU MUST DO TWO (AND ONLY TWO) OF THE QUESTIONS IN EACH AREA.

OF THE TOTAL OF EIGHT QUESTIONS THAT ARE DONE, A MINIMUM OF TWO MUST BE FROM THE 600-LEVEL QUESTIONS.
RULES FOR THE PHYSICS COMPREHENSIVE EXAMINATION

Fall 2000

In order to assure an equitable and fair Comprehensive Examination for all students the following rules will apply to the examination administered by the Department of Physics.

1. The examination will be closed book. Students will not be permitted to bring to the examination room materials such as notes, reference books, calculators, or other aids of any form. The Physics Department will supply copies of the CRC Mathematical Handbook, Schaum's Mathematical Handbook, Table of Functions by Jahnke and Emde, and the NBS Handbook of Mathematical Functions, for the examinees' use should they feel the need for these references during the examination.

   Students who want to use their own mathematical tables and/or handbooks must get permission from the Chairman at least two days before the beginning of the comprehensive. These tables and/or handbooks will remain under the control of the department until after the comp exam but will be available to their owners during the exam.

2. If needed, students will be allowed a short break, not to exceed 5 minutes, to use the restroom facilities during the examination period. While absent from the examination room students may not consult material or other sources of information concerning any matter pertaining to the examination. Students who leave the examination room will turn in all of their examination materials to the proctor before leaving the room. Examination papers will be returned to the students when they reenter the examination room. Students who fail to return within 5 minutes will be considered to have completed that portion of the examination.

Only one student will be permitted to leave the examination room at a time.
Consider a dilute gas of \( N \) molecules confined to a chamber of volume \( V \) where it is in equilibrium at some high temperature \( T \). Through a window in the chamber containing the gas, one observes a spectral line characteristic of some electronic transition. Because of the molecules' thermal motion the observed spectral line is *Doppler broadened*. 

(a) Show that the intensity function describing the shape of the line as a function of frequency \( S(\nu) \) is a Gaussian function centered at \( \omega_0 \), the spectral frequency for a molecule at rest. Treat the translational degrees of freedom classically. 

**HINT:** Recall that the Doppler shifted frequency \( \nu \) for a light source moving toward (+) or away from an observer (-) at a speed \( s \ll c \) is 

\[
\nu = \frac{\nu_0}{1 \pm s/c},
\]

where \( \nu_0 \) is the frequency observed in the rest frame of the source.

(b) Obtain the temperature dependence of the width of the line.

Suppose that the gas is composed of rigid symmetric-top molecules (two equal moments of inertia). The rotational energy levels are given by 

\[
e_{j\lambda m} = \frac{1}{2} \hbar J(j + 1) + \lambda^2 \left( \frac{1}{I_3} - \frac{1}{I_1} \right) = \alpha j(j + 1) + \gamma \lambda^2
\]

where \( I_1, I_4 \) and \( I_3 \) are the principal moments of inertia and where the quantum numbers \( j, \lambda, \) and \( m \) take the values 

\[
0 \leq j < \infty; \quad -j \leq \lambda < j; \quad -j \leq m \leq j.
\]

The energy is degenerate with respect to \( m \). Assume \( I_1 > I_3 \) so that \( \gamma > 0 \).

(c) Continuing to treat the translational motions classically, obtain the total partition function of the system in the form 

\[
Z_N(V, T) = f(N, V, T) \left[ \sum_j \sum_\lambda g_{j\lambda}(\alpha, \beta, \gamma) \right]^N
\]

where \( \beta = 1/kT \) and \( f \) and \( g \) are functions that you must determine. Here \( f(N, V, T) \) is the translational partition function and the other factor is the rotational partition function.

(d) From this obtain the specific heat of the gas in the limit of high temperatures.

**HINT:** For part (d) use the Euler-MacLaurin formula 

\[
\sum_{i=m}^{n} F_i = \int_{m}^{n} dx F(x) + \frac{1}{2} [F(n) + F(m)] - \sum_{k=1}^{\infty} (-1)^k \frac{B_k}{(2k)!} [F^{(2k-1)}(n) - F^{(2k-1)}(m)]
\]

where \( F^{(\mu)} \) denotes the \( \mu \)th derivative of \( F \) and \( B_k \) are the Bernoulli numbers. Retain only the first term on the right and try integrating by parts.
The Debye theory treats a crystal as a continuum and hence the phonon dispersion relation is $\omega = ck$, where $c$ is the speed of sound in the body. In a ferromagnetic solid at low temperatures, there also exist quantized waves of magnetization (magnons) for which the dispersion relation is of the form $\omega = bk^2$, where $b$ is a constant. In both these cases a large-$k$ (small wavelength) cutoff must be imposed; take the cutoffs in the two cases as $k_D$ and $k_M$, respectively.

Obtain expressions valid near $T = 0$ for the phonon and magnon contributions to the specific heat. Your answers may include a dimensionless multiplicative constant in the form of an integral that you need not evaluate. The main point of this problem is to obtain the temperature dependence of the two contributions to the specific heat at low temperatures.
An infinitely long cylinder with radius $a$ and with dielectric constant $\varepsilon$ is placed in an initially uniform electric field $E_0$ which is directed perpendicular to the axis of the cylinder.

(a) Find the potential at all points interior and exterior to the cylinder.

(b) What is the magnitude of the induced dipole per unit length?
A positively charged particle $+q$ sits a distance $d$ away from the center of a conducting spherical shell of radius $R$ where $d < R$. The conducting shell is electrically isolated and holds a total charge $+Q$, which is also positive.

(a) Explain (in words and pictures) how to use superposition to separate this problem into two or more easier ones that you know how to solve.

(b) What is the force on the particle?

(c) Find the value of $Q$ for which the total force on $q$ is equal to zero. Describe the condition for which there is a net attraction between the positive point charge and the positively charged sphere.

HINT: Image charge, grounded sphere of radius $a$:

$$ q' = -\frac{a}{y} q; \quad y' = \frac{a^2}{y} \quad (q, y \rightarrow \text{outside}; \ q', y' \rightarrow \text{inside}) $$
A circular loop of radius $b$, mass $m$, and resistance $R$ rotates about an axis perpendicular to a magnetic field $B$. The axis of rotation passes along a diagonal of the loop. If the angular velocity of the loop is initially $\omega_0$, calculate the time required for the loop to slow to $1/e$ of the initial angular velocity due to resistive heating of the wire.
Consider a plane parallel-plate capacitor with plate area \( A = 200 \text{ cm}^2 \) and plate separation \( d = 2.0 \text{ cm} \). A potential difference \( V_0 = 50 \text{ Volts} \) is supplied by a battery. The space between the plates is initially free space.

(a) What is the charge \( q \) on the capacitor? What is the capacitance \( C_0 \)?

(b) What is the electric field strength \( E_0 \) between the plates? What is the energy stored in the capacitor?

The battery is then disconnected such that the initial charge \( q \) remains on the plates of the capacitor. A dielectric material of dielectric constant \( \kappa = 2.0 \) is then inserted into the capacitor so that it fills the space between the plates.

(c) What is the new capacitance \( C' \) and new voltage \( V' \) of the capacitor?

(d) Based on the figure above, what are the magnitude and direction of

(i) the polarization \( P \),

(ii) the electric displacement \( D \), and

(iii) the electric field \( E \)

with the dielectric present between the plates?

(e) Is work required to remove the dielectric from the capacitor? If so, how much? Explain your answer physically.
Consider two concentric conducting spherical shells. The inner shell has inner radius $a$ and outer radius $b$, while the outer shell has inner radius $c$ and outer radius $d$. The inner shell has a charge $Q_1$ while the outer shell has charge $Q_2$.

(a) Obtain an expression for the surface charge density on each of the 4 conducting surfaces.

(b) What is the capacitance of this set of shells?

(c) If the space between the shells ($b < r < c$) is filled with a nonconducting material with dielectric constant $K$, what then is the charge on each of the 4 conducting surfaces?

(d) What is the polarization charge induced in the dielectric at each surface?

(e) What is the capacitance of the system with the added dielectric?
A 100-g sample of a pure substance is heated to a temperature of 50°C where it is a liquid. It is then removed from the furnace and allowed to cool at constant pressure to room temperature (20°C). Assume that Newton's law of cooling for the rate of heat loss

\[ \frac{dQ}{dt} \propto T - T_{\text{env}} \]

applies. Here \( T_{\text{env}} \) is the temperature of the environment.

The temperature is measured at two-minute intervals after the sample is removed from the furnace and the data at the right are obtained. From these data and the known specific heat of the substance in solid form (0.060 cal/g-K)

(a) determine the melting point of the substance;
(b) determine the latent heat of fusion; and
(c) determine the change of entropy associated with melting.

State explicitly any additional assumptions that you make.

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\[ S = 1.4 \text{ kW/m}^2 \]

and the infrared intensity is

\[ I = b T^4 \]

where \( b = 5.7 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4 \).

(a) By assuming that Earth does not heat up on the average, write down the equation of energy conservation between the incoming and outgoing radiation. Allow for the fact that 30% of the solar radiation is reflected back out to space without being absorbed. Assume that the infrared emission of Earth’s surface is characterized by an average temperature \( T_e \).

HINT: What is the shape of Earth’s cross-section for intercepting solar radiation?

(b) Hence calculate \( T_e \).

A more realistic model takes into account the greenhouse warming effect of the atmosphere. This is done by treating the atmosphere as a spherical shell enclosing Earth. Assume that the shell has the same area as Earth’s surface. It is transparent to the incoming solar radiation but absorbs some of the outgoing infrared radiation from Earth’s surface. It is thus heated and radiates its own infrared radiation, from both its upper and lower surfaces, with intensity \( I = b T_A^4 \) (see diagram).

(c) Write the equation of energy conservation for this shell. Let the temperature of Earth be \( T_e \) and that of the shell be \( T_A \). Assume that the shell absorbs a fraction \( f \) of the outgoing infrared radiation, and that its surface area (inner and outer) is the same as Earth’s.

(d) Use this to express the temperature of the shell \( T_A \) in terms of Earth’s surface temperature \( T_e \).

(e) Write a new equation of energy conservation for Earth’s surface which includes the downward infrared radiation from the shell, which is absorbed by Earth’s surface.

(f) Finally, calculate a new value for the temperature of Earth’s surface \( T_e \). Use \( f = 0.75 \). This new value includes the greenhouse warming.
The moment of inertia tensor can be expressed by

\[ I_g = \int_{\text{vol}} \rho \left[ \delta_{ij} \sum_{k=1}^{3} x_k^2 - x_i x_j \right] d\nu, \]

where \( \rho \) is the density and \( i, j, \) and \( k \), are the indices for the three axes.

(a) Calculate the moment of inertia tensor for a cube of dimension \( a \) for the case where the three axes lie along the edges of the cube.

(b) Use this tensor to calculate the energy of rotation for a cube rotating with an angular velocity \( \omega \) about an axis given by \( x = y \).
Two pendulums with identical lengths \( l \) and masses \( m \) are coupled by a linear spring of spring constant \( k \) (see the sketch at the right). Their motion is confined to the \( x-z \) plane (the plane of the paper).

(a) Write the Lagrangian \( L \) for this system.

(b) Find the equations of motion for the system using

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_i} \right) - \frac{\partial L}{\partial \theta_i} = 0 \quad (i = 1, 2).
\]

(c) Taking the limit of small displacements, find the normal modes and associated frequencies of this two pendulum system. Describe the normal modes.

(d) The system is started at \( t = 0 \) with \( \theta_1(0) = \theta_2(0) = 0 \), \( \dot{\theta}_1(0) = 0 \) and \( \dot{\theta}_2(0) = 3 \) rad/s. Find its subsequent time evolution.
A billiard ball with a radius $R$ and mass $M$ rests on a flat, horizontal table. The ball is struck along its horizontal axis (as shown) so that it moves forward with an initial velocity $v_0$. As the ball moves it experiences sliding friction with a coefficient $\mu$. The friction causes a torque, which increases the rotation rate of the ball until it rolls without slipping.

(a) Draw a diagram showing the forces and torques acting on the ball.
(b) Write down the equations of motion of the ball before it stops slipping.
(c) How long does it take until the ball rolls without slipping?
(d) What is the translational velocity at the time the ball ceases to slip?

HINT: The moment of inertia of a sphere is $\frac{2}{5}MR^2$. 
A particle of mass $m$ moves under the influence of an attractive central force $f(r)$.

(a) Show that by a proper choice of the initial velocity a circular orbit of radius $r_0$ a circular orbit will result.

(b) The circular orbit is subjected to a small radial perturbation. Determine the relationship that must hold between $f(r)$ and $df/dr$ for the orbit to be stable.

(c) If the force law is of the form $f(r) = -\frac{K}{r^n}$ (with both $K$ and $n$ positive constants), determine the maximum value of $n$ for which a circular orbit is stable.

**HINT:** It may help to make a sketch of the effective potential $V_{\text{eff}}(r)$ versus $r$. To obtain the effective potential try writing the angular momentum of the system and then substitute this in the equation for the total energy of the system.
Tritium, $^3_1\text{H}$, a nucleus with one proton and two neutrons, decays via beta emission to helium-3, $^3_2\text{He}$, a nucleus with two protons and one neutron. If the electron bound to the tritium nucleus is in the ground state before the decay, what is the probability that the electron will be in the ground state of the $^3_2\text{He}$ atom after the decay?

The radial wave function for the ground state of a one-electron atom has the form $e^{-2r/a_0}$ where $Z$ is the atomic number, $r$ is the radial coordinate, and $a_0$ is the Bohr radius. Ignore the mass difference between the helium-3 and tritium nuclei.

Mathematical hint: $$\int_0^\infty dx \; x^n e^{-kx} = \frac{n!}{k^{n+1}}$$
Consider a particle of mass $m$ confined to a spherical well of radius $a$:

$$V(r) = \begin{cases} 
0 & (r < a) \\
\infty & (r \geq a)
\end{cases}$$

This might describe a particle confined in a spherical quantum dot.

(a) Begin with the time-dependent Schrödinger equation and separate out the time dependence to get the time-independent Schrödinger equation for the stationary states of the system.

(b) Again use separation of variables to derive separate differential equations for the radial and angular dependences of the eigenfunction. Show that the radial equation is

$$\frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} + \left[ \ell^2 - \frac{\ell(\ell + 1)}{r^2} \right] R(r) = 0$$

where $R(r)$ is the radial function. Identify the constant $\ell^2$.

(c) The solutions to this are the spherical Bessel functions $j_\ell(kr)$ and the spherical Neumann functions $n_\ell(kr)$, the first few of which are given below. One of these must be discarded on physical grounds. Which one? Explain.

(d) Obtain the condition that determines the energy eigenvalues and use it to obtain an equation for the energies. Use this to calculate the ground state energy.

The first few spherical Bessel and Neumann functions are

$$j_0(x) = \frac{\sin x}{x}; \quad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}; \quad j_2(x) = \left( \frac{3}{x^3} - \frac{1}{x} \right) \sin x - \frac{3 \cos x}{x^2}$$

and

$$n_0(x) = -\frac{\cos x}{x}; \quad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}; \quad n_2(x) = -\left( \frac{3}{x^3} - \frac{1}{x} \right) \cos x - \frac{3 \sin x}{x^2}.$$
A particle of mass $m$ moves under the influence of the one-dimensional potential

$$V(x) = \begin{cases} 
-V_0 \left( 1 - \frac{|x|}{a} \right) & (x \leq a) \\
0 & (x > a) 
\end{cases}$$

(a) Use the WKB approximation to find the energy levels of the bound states.

(b) Under what conditions will the WKB method yield a good approximation to the exact energy levels?
Use time-dependent perturbation theory to calculate the ground state and first excited state energies for a one-dimensional system involving a single particle of mass $m$ moving under the influence of a potential energy
\[ V(x) = \frac{1}{2} m \omega^2 x^2 + \lambda x^4, \]
where $\omega$ is the classical oscillation frequency and $\lambda$ is a constant.

**Eigenfunctions for the harmonic oscillator:**

\[ \psi_0 = (a/\pi)^{\lambda/4} e^{-ax^2}, \]
\[ \psi_1 = (4a^3/\pi)^{\lambda/4} xe^{-ax^2}, \]
\[ \psi_2 = (a/4 \pi)^{\lambda/4} (2ax^2 - 1) e^{-ax^2/2}, \]
where $a = m \omega / \hbar$. 

Consider a high energy x-ray photon that is scattered by a free electron (rest mass \( m_e \); \( m_e c^2 = 0.511 \text{ MeV} \)) that is initially at rest. Assume that the x-ray photon’s energy is of the same order of magnitude as \( m_e c^2 \).

What is the difference in wavelengths between the outgoing and incident photons for scattering through an angle \( \theta \) with respect to the direction of travel of the incident photon?